Arbitrary-norm Support Vector Machine. Properties and Applications

Emilio Carrizosa
Universidad de Sevilla, Spain
http://ecarriz.us.es

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Research project from Spanish Science Ministry

- Rafael Blanquero
- Emilio Carrizosa
- Eduardo Conde
- José Gordillo
- Belén Martín-Barragán
- Frank Plastria
- Dolores Romero Morales

Support Vector Machines: A Data Mining tool

In between many fields

- Machine Learning / Statistics
- O.R. ...

With many applications ...

- Bioinformatics (gene expression, ...) (e.g. 327 papers have the term "Support Vector Machines" in title/abstract in Bioinformatics
- CRM
- Banking (credit scoring, credit card fraud detection, ...)
- Internet (spam detection, ...)
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Supervised classification (Discriminant analysis)

- Population: $\mathcal{O}$
- $u \in \mathcal{O} \hookrightarrow \begin{cases} x_u \in \mathbb{R}^N & (\text{ features }) \\ y_u \in \mathcal{C} & (\text{ class labels }) \end{cases}$
- $I \subset \mathcal{O}$: *training sample*: individuals u for which the pair (x_u, y_u) *is known*
- Aim: build a classification rule $\varphi : \mathbb{R}^N \longmapsto \mathcal{C}$, i.e., build a function assigning labels to objects u for which just x_u (and not y_u!) is known

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Fisher, 1936

Classification with 2 labels

- $I \neq \emptyset$, $|I| < \infty$. $I = I_+ \cup I_-$, with $I^+ \cap I^- = \emptyset$, $I_+, I_- \neq \emptyset$.
- $i \in I \hookrightarrow \begin{cases} x_i \in \mathbb{R}^N \\ y_i := \begin{cases} 1, & \text{if } i \in I_+ \\ -1, & \text{if } i \in I_- \end{cases} \end{cases}$

For $\omega \in \mathbb{R}^N$, $\beta \in \mathbb{R}$

$$x \in \mathbb{R}^N \mapsto \begin{cases} 1, & \text{if } \omega^\top x + \beta > 0 \\ -1, & \text{if } \omega^\top x + \beta < 0 \end{cases}$$

We force $\omega \in \Omega := \left\{\omega \in \mathbb{R}^N : q_j^\top \omega \geq 0 \, \forall j \in J\right\}$, for a finite set $\{q_j : j \in J\}$

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Can we do it right, i.e. score 100%. . . . . . at least on I?

$(\{\omega_i : i \in I_+\}, \{\omega_i : i \in I_-\})$: *Ω-separable* if $\exists(\omega, \beta) \in \Omega \times \mathbb{R}$:
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Such (ω, β) : *Ω-separating* $(\{x_i : i \in I_+\}, \{x_i : i \in I_-\})$.

Ω- Separability: equivalent to

$$(conv(\{x_i : i \in I_+\}) + cone(\{q_j : j \in J\})) \cap conv(\{x_i : i \in I_-\}) = \emptyset$$

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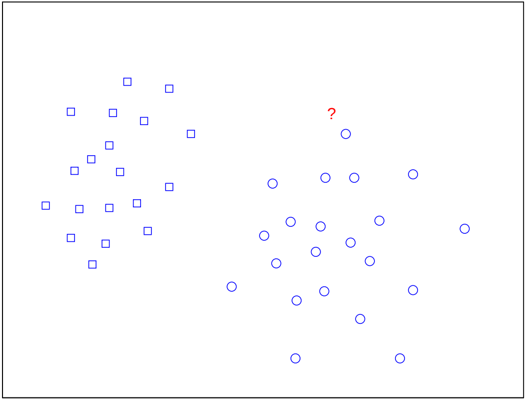
How to chose one linear classifier?



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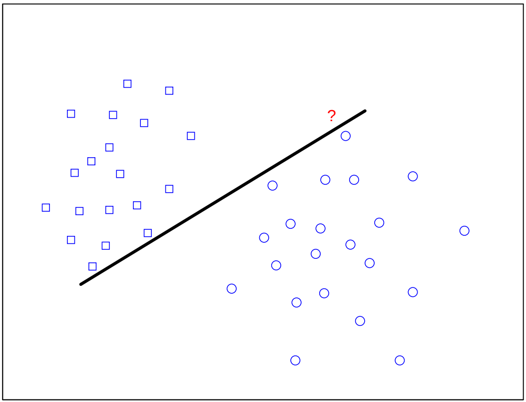

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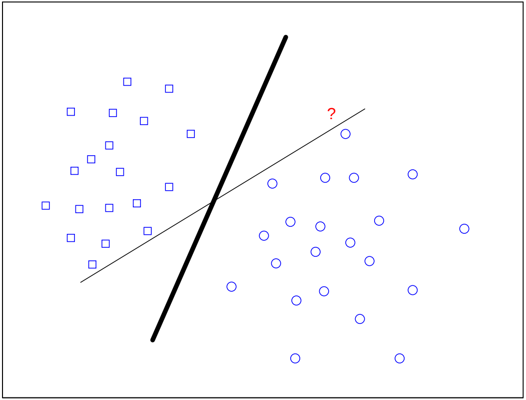

Problem statement

Halfspace of misclassification

- for $i \in I_+ : H(\omega, \beta)^- := \{x \in \mathbb{R}^N : \omega^\top x + \beta \leq 0\}$
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- $\delta_i(\omega, \beta) :=$ distance (measured by some norm $\|\cdot\|$) from x_i to halfspace of misclassification
- *margin*: $\min_{i \in I} \delta_i(\omega, \beta)$

Maximize the margin

$$\max_{\omega \in \Omega \setminus \{0\}} \left(\min_{i \in I} \delta_i(\omega, \beta) \right)$$

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$$\begin{array}{ll} \max & \min_{i \in I} \frac{y_i(\omega^\top x_i + \beta)}{\|\omega\|^\circ} \\ \text{s.t.} & \omega \in \Omega \setminus \{0\} \end{array}$$

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$$\begin{array}{ll} \min & \omega^\top \omega \\ \text{s.t.} & y_i(\omega^\top x_i + \beta) \geq 1 \, \forall i \in I \\ & \omega \in \Omega \end{array}$$

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$$\begin{array}{lll} \min & t & \\ \text{s.t.} & y_i\left(\omega^\top x_i + \beta\right) \geq 1 & \forall i \in I \\ & -t \leq \omega_k \leq t & \forall k = 1, 2, \ldots, N \\ & \omega \in \Omega & \end{array}$$

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$$\begin{array}{lll} \min & e^\top \omega_+ + e^\top \omega_- & \\ \text{s.t.} & y_i\left(\omega_+^\top x_i - \omega_-^\top x_i + \beta\right) \geq 1 & \forall i \in I \\ & \omega_+, \omega_- \geq 0 & \\ & \omega_+ - \omega_- \in \Omega & \end{array}$$

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... The advantages of using polyhedral norms ...

Cost of getting info

- π_k cost associated with k-th variable
- Total cost

$$\Pi(\omega, \beta) = \sum_{k:\, \omega_k \neq 0} \pi_k$$

Objective

$$\{\max \min_i \delta_i(\omega, \beta);\ \min \Pi(\omega, \beta)\}$$

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Formulation

$$
\begin{array}{ll}
\max & y, \quad \min \sum_{k=1}^{N} \pi_k z_k \\
\text{s.t.:} & \sum_{k=1}^{N} \phi_k(x^u)\left(\alpha^i_{+k} - \alpha^i_{-k} - \alpha^j_{+k} + \alpha^j_{-k}\right) + \beta^i - \beta^j - y \geq 0, \quad \forall i \neq ; \; u \in I_i \\
& \sum_{c=1}^{C} \sum_{k=1}^{N} \left(\alpha^c_{+k} + \alpha^c_{-k}\right) \leq 1 \\
& \sum_{h:k \preceq h} \sum_{c=1}^{C} \left(\alpha^c_{+k} + \alpha^c_{-k}\right) \leq z_k, \quad \forall k = 1, 2, \ldots, N \\
& \alpha^c_{+k} \geq 0 \quad \forall k = 1, 2, \ldots, N; \; c = 1, 2, \ldots, C \\
& \alpha^c_{-k} \geq 0 \quad \forall k = 1, 2, \ldots, N; \; c = 1, 2, \ldots, C \\
& \beta^c \geq 0 \quad \forall c = 1, 2, \ldots, C \\
& z_k \in \{0, 1\} \quad \forall k = 1, 2, \ldots, N
\end{array}
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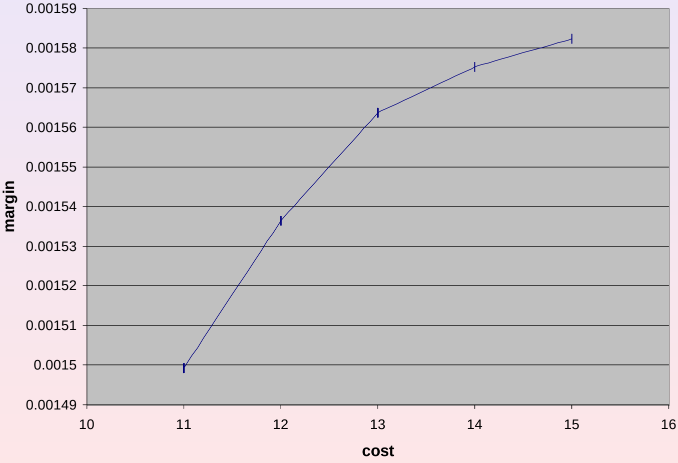
Supported efficient solutions



- Separability conditions: rather strong. Still, model important since
  - Cornerstone for more sophisticated models (valid also for non-separable sets)
  - Separability may happen e.g. in small-size high-dimensional data sets
- Let's address first the Ω-separable case ...

- For $\omega \in \mathbb{R}^N \setminus \{0\}, \beta_+ \leq \beta_-$, define the band

$$\mathcal{B}(\omega, \beta_-, \beta_+) = \{x \in \mathbb{R}^N : \omega^\top x + \beta_+ \leq 0 \leq \omega^\top x + \beta_-\}$$

- Its width $W(\mathcal{B}(\omega, \beta_-, \beta_+))$:

$$\begin{aligned} W(\mathcal{B}(\omega, \beta_-, \beta_+)) &= \min \quad \|x_+ - x_-\| \\ &\quad \text{s.t.} \quad \begin{array}{lcl} x_- & \in & H(\omega, \beta_-)^- \\ x_+ & \in & H(\omega, \beta_+)^+ \end{array} \\ &= \frac{\beta_- - \beta_+}{\|\omega\|^\circ} \end{aligned}$$

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$$\begin{aligned} \Delta &= \frac{\beta_- - \beta_+}{2} \\ \beta &= \frac{\beta_- + \beta_+}{2} \end{aligned}$$

$$\begin{array}{lll} \max & \frac{\Delta}{\|\omega\|^\circ} & \\ \text{s.t.} & \omega^\top x_i + \beta \leq -\Delta & \forall i \in I_- \\ & \omega^\top x_i + \beta \geq \Delta & \forall i \in I_+ \\ & \omega \in \Omega, \Delta \geq 0 & \end{array}$$

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$$\begin{array}{ll} \max & W(\mathcal{B}(\omega,\beta_-,\beta_+)) \\ \text{s.t.} & \{x_i : i \in I_-\} \subset H(\omega,\beta_-)^- \\ & \{x_i : i \in I_+\} \subset H(\omega,\beta_+)^+ \\ & \omega \in \Omega \end{array}$$

$$\begin{array}{lll} \max & \frac{\beta_-\beta_+}{\|\omega\|^\circ} & \\ \text{s.t.} & \omega^\top x_i + \beta_- \leq 0 & \forall i \in I_- \\ & \omega^\top x_i + \beta_+ \geq 0 & \forall i \in I_+ \\ & \beta_+ \leq \beta_-, \omega \in \Omega & \end{array}$$

$$\begin{array}{rcl} \Delta & = & \dfrac{\beta_- - \beta_+}{2} \\ \beta & = & \dfrac{\beta_- + \beta_+}{2} \end{array}$$

$$\begin{array}{lll} \max & \frac{\Delta}{\|\omega\|^\circ} & \\ \text{s.t.} & \omega^\top x_i + \beta \leq -\Delta & \forall i \in I_- \\ & \omega^\top x_i + \beta \geq \Delta & \forall i \in I_+ \\ & \omega \in \Omega, \Delta \geq 0 & \end{array}$$

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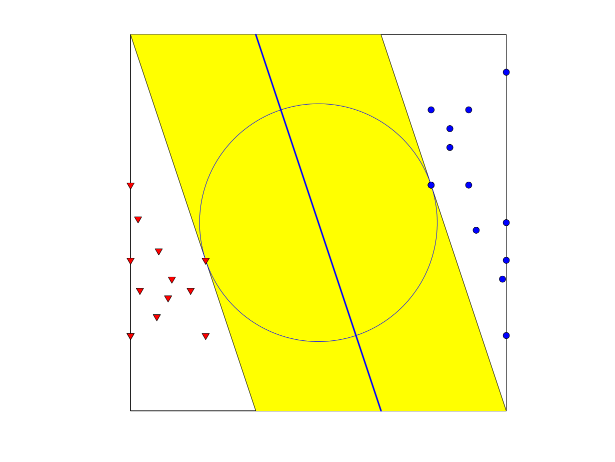
$$

Maximal width band



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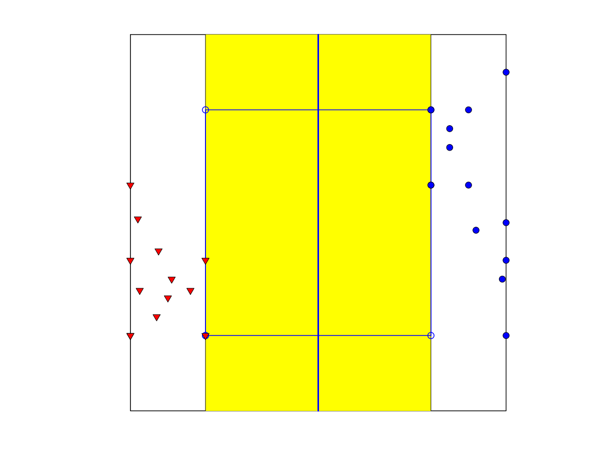


Existence & uniqueness

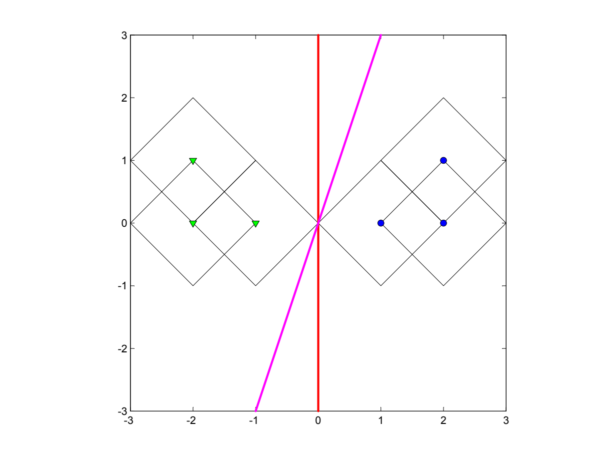
$$\begin{array}{ll} \min & \|\omega\|^{\circ} \\ \text{s.a.} & y_i(\omega^{\top} u_i + \beta) \geq 1 \, \forall i \in I \\ & \omega \in \Omega \end{array} \qquad (P)$$

Assumed to be feasible:

$$(conv(\{u_i \, : \, i \in I_+\}) + cone(\{q_j \, : \, j \in J\})) \cap conv(\{u_i \, : \, i \in I_-\}) = \emptyset$$

Th: Set $\mathcal{S}(P)$ of optimal solutions: non-empty. If $\| \cdot \|$: smooth, then $|\mathcal{S}(P)| = 1$

Uniqueness: ?



Optimality conditions

Th: Let (ω^*, β^*) : feasible for (P). Following statements are equivalent:

1. $(\omega^*, \beta^*) \in \mathcal{S}(P)$.
2. $\exists \lambda \in \mathbb{R}^I$, $\mu \in \mathbb{R}^J$ s.t.

$$
\begin{array}{rcll}
\lambda, \mu & \geq & 0 & \\
\sum_{i \in I} \lambda_i y_i & = & 0 & \\
\sum_{i \in I} \lambda_i y_i x_i + \sum_{j \in J} \mu_j q_j & \in & \partial \|\omega^*\|^\circ & \\
\lambda_i \left(y_i \left({\omega^*}^\top x_i + \beta^* \right) - 1 \right) & = & 0 & \forall i \in I \\
\mu_j \left({\omega^*}^\top q_j \right) & = & 0 & \forall j \in J
\end{array}
$$

Optimality conditions (Euclidean norm)

Th: Let (ω^*, β^*) : feasible for (P) and $\|\cdot\|$: Euclidean norm.
Following statements are equivalent:

1. $(\omega^*, \beta^*) \in \mathcal{S}(P)$.
2. $\exists \lambda \in \mathbb{R}^I$, $\mu \in \mathbb{R}^J$ s.t.

$$\begin{array}{rcll} \lambda, \mu & \geq & 0 & \\ \sum_{i \in I} \lambda_i y_i & = & 0 & \\ \sum_{i \in I} \lambda_i y_i x_i + \sum_{j \in J} \mu_j q_j & = & \omega^* & \\ \lambda_i \left(y_i \left({\omega^*}^\top x_i + \beta^* \right) - 1 \right) & = & 0 & \forall i \in I \\ \mu_j \left({\omega^*}^\top q_j \right) & = & 0 & \forall j \in J \end{array}$$

Support vectors

$$I_+(\lambda) = \{i \in I_+ : \lambda_i > 0\} \quad I_-(\lambda) = \{i \in I_- : \lambda_i > 0\}$$

$$\begin{array}{ll} I(\lambda) = I_+(\lambda) \cup I_-(\lambda) & \{x_i : \in I(\lambda)\} : \text{ support vectors at } \lambda \\ J(\mu) = \{j \in J : \mu_j > 0\} & \{q_j : j \in J(\mu)\} : \text{ support normals at } \mu \end{array}$$

Th: Let $(\omega^*, \beta^*) \in \mathcal{S}(P)$. For each $\lambda \in \mathbb{R}^I$, $\mu \in \mathbb{R}^J$, multipliers in (ω^*, β^*),

1. $I_+(\lambda) \neq \emptyset$, $I_-(\lambda) \neq \emptyset$
2. (ω^*, β^*) : also optimal for

$$\begin{array}{lll} \min & \|\omega\|^\circ & \\ \text{s.t.} & y_i(\omega^\top x_i + \beta) & \geq 1 \quad \forall i \in I(\lambda) \\ & \omega^\top q_j & \geq 0 \quad \forall j \in J(\mu). \end{array}$$

Support vectors

Th: Given $(\omega^*, \beta^*) \in \mathcal{S}(P)$, $\nexists \beta \neq \beta^*$ s.t. $(\omega^*, \beta) \in \mathcal{S}(P)$.

Th: Let $(\omega^*, \beta^*) \in \mathcal{S}(P)$. $\exists \lambda \in \mathbb{R}^I$, $\mu \in \mathbb{R}^J$, multipliers at (ω^*, β^*), s.t. $|I(\lambda) \cup J(\mu)| \leq N + 1$

Th: Let $(\omega^*, \beta^*) \in \mathcal{S}^*$, and let (λ^*, μ^*) : multiplier at (ω^*, β^*).
Given (ω, β), feasible for (P), following statements are equivalent:

1. $(\omega, \beta) \in \mathcal{S}(P)$
2. (ω, β) satisfies

$$\begin{array}{rcl} \sum_{i \in I} \lambda_i^* y_i x_i + \sum_{j \in J} \mu_j^* q_j & \in & \partial \|\omega\|^\circ \\ y_i \left(\omega^\top x_i + \beta\right) & = & 1 \;\; \forall i \in I(\lambda^*) \\ \omega^\top q_j & = & 0 \; \forall j \in J(\mu^*) \end{array}$$

3. (ω, β) satisfies:

$$\begin{array}{rcl} \omega & \in & N_B(\sum_{i \in I} \lambda_i^* y_i x_i + \sum_{j \in J} \mu_j^* q_j) \\ y_i \left(\omega^\top x_i + \beta\right) & = & 1 \;\; \forall i \in I(\lambda^*) \\ \omega^\top q_j & = & 0 \; \forall j \in J(\mu^*) \end{array}$$

$$
\begin{aligned}
\|u\|_\Omega &:= \max\left\{u^\top x : \|x\|^\circ \leq 1,\ x \in \Omega\right\}\\
&= \min\left\{\|u+q\| : q \in cone(\{q_j : j \in J\})\right\}
\end{aligned}
$$

Dual

$$
\begin{array}{ll}
\min & \|\sum_{i\in I} \mu_i y_i u_i\|_\Omega \\
\text{s.t.} & \sum_{i\in I} \mu_i y_i = 0 \\
& \sum_{i\in I} \mu_i = 1 \\
& \mu \geq 0.
\end{array}
\qquad (D)
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- Optimal values of (P), (D): coincide
- A maximal margin hyperplane can be obtained from optimal dual solution

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- Optimal values of (P), (D): coincide
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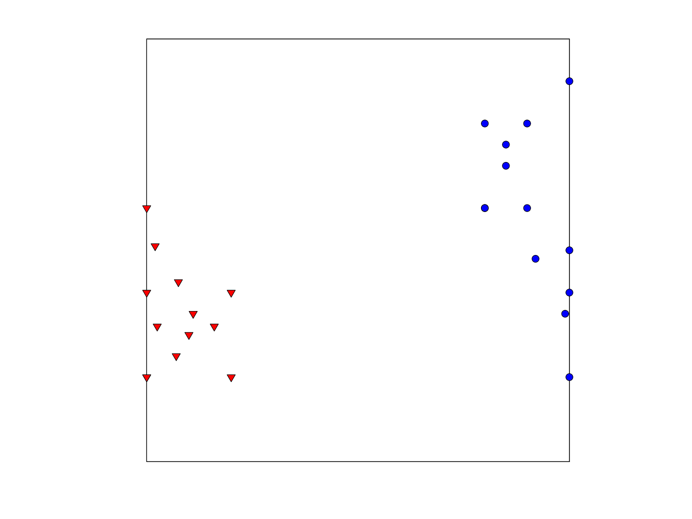
Duals & closest pairs

$$
\begin{array}{ll}
\min & \|a_+ + q - a_-\| \\
\text{s.a.} & a_+ \in conv\left(\{u_i : i \in I_+\}\right) \\
& a_- \in conv\left(\{u_i : i \in I_-\}\right) \\
& q \in cone\left(\{q_j : j \in J\}\right).
\end{array}
$$

- Euclidean unconstrained case:
  - $q^* = 0$
  - $a_+^* - a_-^*$: direction of maximal margin hyperplane
- In general, $a_+^* - a_-^*$: subgradient of x^*, of optimal direction

Dual & closest pairs



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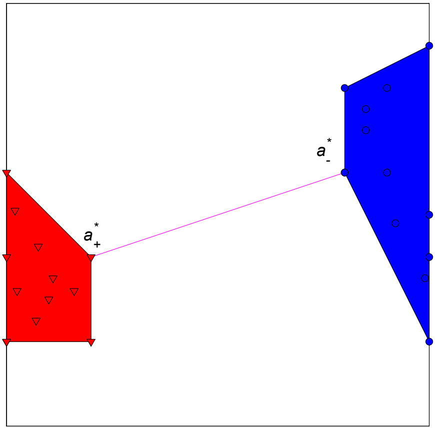

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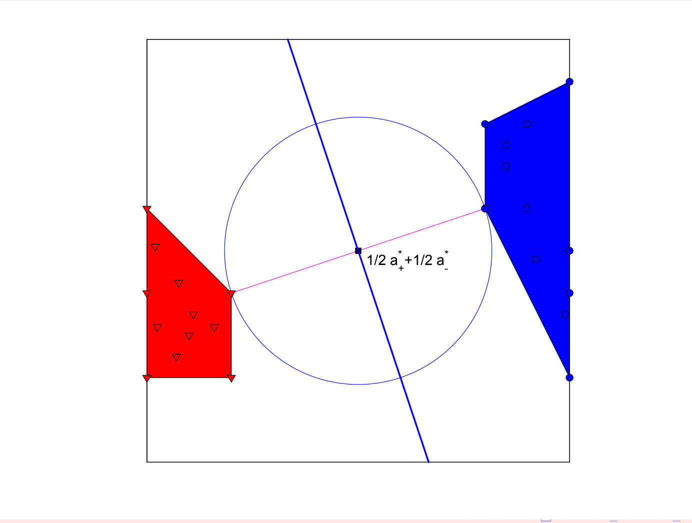

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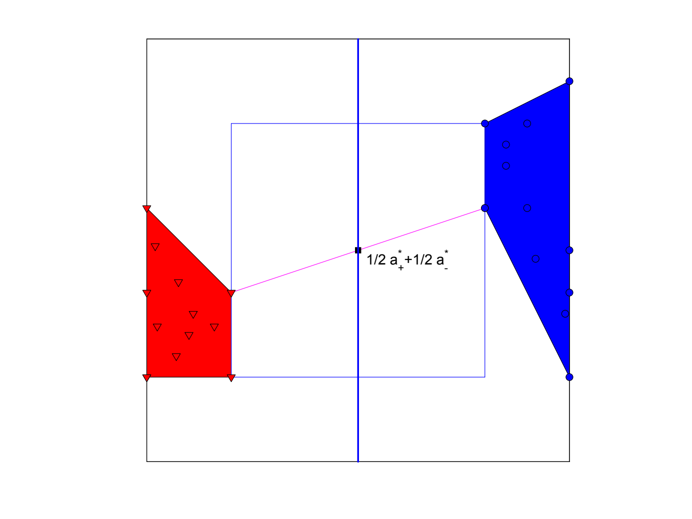

Two margins at same time

Margin of class c

$$\rho^c(\omega,\beta) = \min_{i\in I_c} \frac{y_i(\omega^\top x_i + \beta)}{\|\omega\|^\circ}$$

Problem:

Simultaneous maximization of both margins

$$\max\{\rho^1(\omega,\beta),\ \rho^{-1}(\omega,\beta)\}$$

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Key result

For $\|\cdot\|$: smooth, all efficient solutions anterior correspond to parallel hyperplanes.

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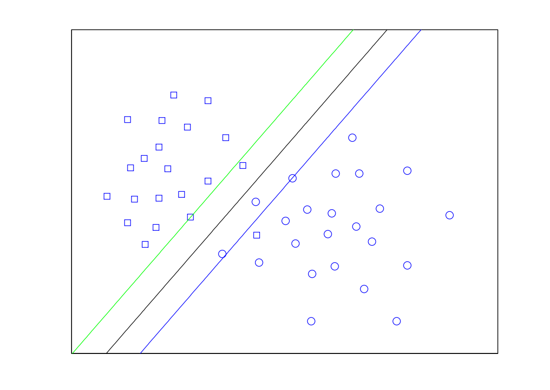
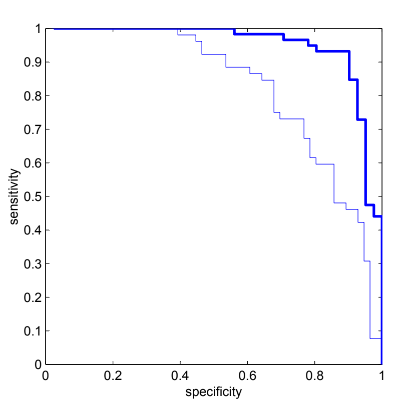


Choice of β.

ROC curve



Non-separable populations

All the above, valid just when $\{x_i : i \in I_+\}, \{x_i : i \in I_-\}$: Ω-separable. When not the case, (P) : unfeasible, (D) : unbounded.

Possible strategies

1. change the objective function into something not requiring separability
2. modify the data, coming up with separable groups:
    - *Value embedding*
    - *Label embedding: soft margin*
3. (of course) all the above together

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Value embedding

Take $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}^T$, and replace each (x_i, y_i) by $(\varphi(x_i), y_i)$

"Linear" classifier sought:

$$\omega^\top \varphi(x) + \beta \qquad (\omega \in \mathbb{R}^T \beta \in \mathbb{R})$$

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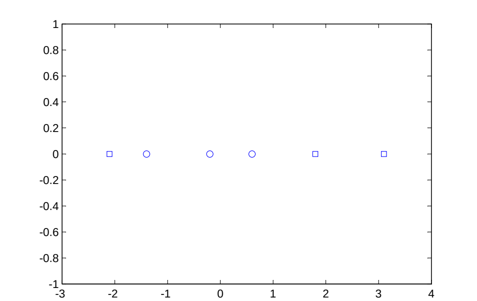

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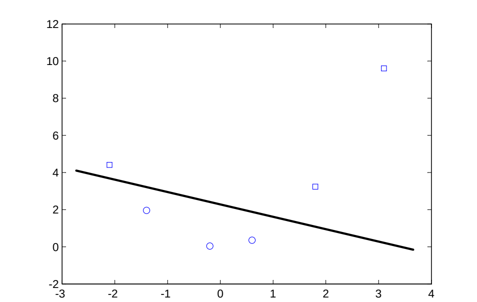

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Example I

$$\begin{aligned} \varphi(s_1, s_2, \ldots, s_N) \quad = \quad & (s_1, s_2, \ldots, s_N, \\ & s_1^2, s_1 s_2, \ldots, s_1 s_N, \\ & s_2^2, \ldots, s_2 s_N, \\ & \ldots s_N^2) \end{aligned}$$

"linear" classifier of form $x^\top W x + \beta$

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Example II (patterns)

$$\begin{aligned} \tau_{i1} > \tau_{i2} > \ldots > \tau_{ir} \quad &: \quad \text{given } \forall i \\ \varphi(s_1, s_2, \ldots, s_N) &= (\delta(s_i - \tau_{ik}))_{ik} \\ \delta(s_i - \tau_{ik}) &= \begin{cases} 1, & \text{if } s_i - \tau_{ik} > 0 \\ 0, & \text{else} \end{cases} \end{aligned}$$

Combining the two above,

database	$\|\mathcal{O}\|$	N
sonar	208	60
ionosphere	351	34
wdbc	569	30

CART		
database	training	testing
sonar	0.98	0.77
ionosphere	0.98	0.88
wdbc	0.99	0.94

SVM		
database	training	testing
sonar	1.00	0.84
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Kernels (Euclidean norm)

$$\begin{array}{ll} \max & \sum_{i\in I}\lambda_i - \frac{1}{2}\|\sum_i \lambda_i y_i x_i\|^2 \\ \text{s.t.} & \sum_{i\in I}\lambda_i y_i = 0, \quad \lambda \geq 0 \end{array}$$

$$(x_i, y_i) \xrightarrow{\varphi} (\varphi(x_i), y_i)$$

$$\begin{array}{ll} \max & \sum_{i\in I}\lambda_i - \frac{1}{2}\sum_{i,j\in I}\lambda_i\lambda_j y_i y_j \varphi(x_i)^\top \varphi(x_j) \\ \text{s.t.} & \sum_{i\in I}\lambda_i y_i = 0, \quad \lambda \geq 0 \end{array}$$

$$\omega = \sum_i \lambda_i y_i \varphi(x_i)$$

Classification rule:

$$\begin{array}{lll} \text{assign } x \text{ to group 1} & \text{iff} & \omega^\top \varphi(x) + \beta > 0 \\ & \text{iff} & \sum_i \lambda_i y_i \varphi(x_i)^\top \varphi(x) + \beta > 0 \end{array}$$

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Soft margin. Cortes-Vapnik

Replaces (P) by one in the form

$$\begin{array}{ll} \min & \|\omega\|_2^2 + C\sum_{i\in I}\eta_i^p \\ \text{s.t.} & y_i\left(\omega^\top x_i + \beta\right) \geq 1 - \eta_i \,\forall i \in I \\ & \eta \geq 0 \end{array}$$

for $p \geq 1$ (typically $p = 1$) and $C > 0$, or

$$\begin{array}{ll} \min & \|\omega\|_1 + C\sum_{i\in I}\eta_i \\ \text{s.t.} & y_i\left(\omega^\top x_i + \beta\right) \geq 1 - \eta_i \,\forall i \in I \\ & \eta \geq 0. \end{array}$$

Value embedding

- Soft margin: almost particular cases
- enables us to use directly results of the separable case

$$(x_i, y_i) \longrightarrow (\widehat{x}_i, y_i) := (\overbrace{(x_i; 0, \overset{(i)}{y_i}, 0, \ldots, 0)}^{\widehat{x}}, y_i) \in (\mathbb{R}^N \times \mathbb{R}^I) \times \{-1, 1\}$$

Sets $\{\widehat{x}_i : i \in I_+\}$ y $\{\widehat{x}_i : i \in I_-\}$: separable.

Any method applicable to the separable case can be used to the non-separable case after embedding:

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Value embedding

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- enables us to use directly results of the separable case

$$(x_i, y_i) \longrightarrow (\widehat{x_i}, y_i) := (\overbrace{(x_i; 0, \overset{(i)}{y_i}, 0, \ldots, 0)}^{\widehat{x}}, y_i) \in (\mathbb{R}^N \times \mathbb{R}^I) \times \{-1, 1\}$$

Sets $\{\widehat{x_i} : i \in I_+\}$ y $\{\widehat{x_i} : i \in I_-\}$: separable.

Any method applicable to the separable case can be used to the non-separable case after embedding:

$$\begin{array}{ll} \min & \|\widehat{\omega}\|^\circ \\ \text{s.t.} & y_i\left(\widehat{\omega}^\top \widehat{x_i} + \beta\right) \geq 1 \quad \forall i \in I \end{array}$$

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Assumptions

$$\|(\omega, \eta)\| = \gamma(\gamma_1(\omega), \gamma_2(\eta)),$$

- γ_1 : norm in $\mathbb{R}^N$
- γ_2 : norm in $\mathbb{R}^I$, monotonic in $\mathbb{R}^I_+$
- $\gamma(t_1, t_2) = \max\{|t_1|, \frac{1}{C}|t_2|\}$, thus $\gamma^\circ(t_1, t_2) = |t_1| + C|t_2|$

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Primal (P_{soft})	
min	$\gamma_1^\circ(\omega) + C\gamma_2^\circ(\eta)$
s.t.	$y_i\left(\omega^\top x_i + \beta\right) \geq 1 - \eta_i \quad \forall i \in I$

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Primal (P_{soft})	
min	$\gamma_1^\circ(\omega) + C\gamma_2^\circ\left(\left(1 - y_i\left(\omega^\top x_i + \beta\right)\right)^+_{i\in I}\right)$
s.t.	no constraints!

Assumptions

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Dual (D_{soft})	
min	$\max(\gamma_1(\sum_{i\in I}\mu_i y_i x_i), C\gamma_2(\mu))$
s.t.	$\sum_{i\in I}\mu_i y_i = 0$
	$\sum_{i\in I}\mu_i = 1$
	$\mu \geq 0$

Primal (P_{soft})	
min	$\gamma_1^\circ(\omega) + C\gamma_2^\circ(\eta)$
s.t.	$y_i\left(\omega^\top x_i + \beta\right) \geq 1 - \eta_i \quad \forall i \in I$

Proposition

Let $(\omega^*, \eta^*, \beta^*)$, feasible for (P_{soft}). Following statements are equivalent:

1. $(\omega^*, \eta^*, \beta^*)$: optimal for (P_{soft})
2. $\exists \lambda \in \mathbb{R}^I$ s.t.

$$\begin{cases} \lambda & \geq & 0 \\ \sum_{i\in I} \lambda_i y_i & = & 0 \\ \sum_{i\in I} \lambda_i y_i x_i & \in & \partial\gamma_1^\circ(\omega^*) \\ \lambda & \in & C\partial\gamma_2^\circ(\eta^*) \\ \lambda_i\left(y_i\left({\omega^*}^\top x_i + \beta^*\right) - 1 + \eta_i^*\right) & = & 0 \quad \forall i \in I \end{cases}$$

Primal (P_{soft})		
min	$\gamma_1^\circ(\omega) + C\gamma_2^\circ(\eta)$	
s.t.	$y_i\left(\omega^\top x_i + \beta\right) \geq 1 - \eta_i$	$\forall i \in I$

Existence: ok

Proposition

Assume γ_1 is smooth. Let $(\omega^*, \eta^*, \beta^*)$: optimal to (P_{soft}). Any optimal solution to (ω, η, β) satisfies

$$\omega = \nu\omega^*$$

for some $\nu > 0$. In other words, all optimal solutions to (P_{soft}) generate parallel hyperplanes in $\mathbb{R}^N$.

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Degenerate case: $\omega^* = 0, \eta^* \neq 0, \beta^* = 0$

Proposition

Following statements are equivalent:

1. $\exists \eta^* \in \mathbb{R}^I$ s.t. $(0, \eta^*, 0)$: optimal to (P_{soft})
2. $\exists \lambda \in \mathbb{R}^I$ s.t.

$$
\begin{array}{rcl}
\lambda & \geq & 0 \\
\sum_{i \in I} \lambda_i y_i & = & 0 \\
\lambda & \in & \partial \gamma_2^\circ(e) \\
\gamma_1(\sum_{i \in I} \lambda_i y_i x_i &) \leq & C
\end{array}
$$

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Proposition

Following statements are equivalent:

1. $\forall C > 0 \ \exists$ optimal solution to (P_{soft}) in form $(0, \eta, 0)$
2. $\exists \lambda \in \mathbb{R}^I$ s.t.

$$\begin{aligned} \lambda &\geq 0 \\ \sum_{i \in I} \lambda_i y_i &= 0 \\ \lambda &\in \partial\gamma_2^\circ(e) \\ \sum_{i \in I} \lambda_i y_i x_i &= 0 \end{aligned}$$

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Proposition

Assume γ_2 : symmetrical and γ_2° : differentiable at e. Following statements are equivalent:

1. For all $C > 0$, optimal to (P_{soft}) has form $(0, \eta, 0)$
2. $\sum_{i\in I} y_i = 0$, $\sum_{i\in I} y_i x_i = 0$

In practice we can easily avoid trivial solution $\omega = 0, \beta = 0$: drop just one element from database! (with this, $\sum_{i\in I} y_i \neq 0$)

Time for commercials

Dirección http://www.euro2006.org/

EURO XXI in Iceland
July 2-5, 2006

21st European Conference on Operational Research

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- Registration
- Call for Papers
- Abstract Submission
- Program Schedule
- Awards
- Conference Venue
- Exhibition
- Sponsors
- Social Events
- Travel
- Accommodation
- Iceland

List of currently accepted streams (still preliminary):

Stream	Organizer
Adaptive Memory Programming	Cesar Rego (crego@bus.olemiss.edu)
Behavioural and Experimental Economics	Bernd Brandl (bernd.brandl@univie.ac.at) Johannes Leitner (johannesleitner@gmx.de)
Business Sessions	Bjarni Kristjánsson (bjarni@maximalsoftware.com)
Combinatorial Optimization	Silvano Martello (smartello@deis.unibo.it)
Complex Societal Problems	Dorien DeTombe (DeTombe@lri.jur.uva.nl)
Computational Biology and Medicine	Jacek Blazewicz (Jacek.Blazewicz@cs.put.poznan.pl) Metin Türkay (mturkay@ku.edu.tr)
Convex Optimization Methods	Marc Teboulle (teboulle@post.tau.ac.il)
Cutting and Packing	José Fernando Oliveira (jfo@fe.up.pt)
Data Mining	Emilio Carrizosa (ecarrizosa@us.es) Jacob Kogan (kogan@math.umbc.edu)

Decision tables

$\mathcal{D}$	Crit_1	Crit_2	$\ldots$	Crit_N
d_1	$\psi_1(d_1)$	$\psi_2(d_1)$	$\ldots$	$\psi_N(d_1)$
d_2	$\psi_1(d_2)$	$\psi_2(d_2)$	$\ldots$	$\psi_N(d_2)$
$\vdots$	$\vdots$		$\ddots$	$\vdots$
d_m	$\psi_1(d_m)$	$\psi_2(d_m)$	$\ldots$	$\psi_N(d_m)$

Aim

Associate with criteria scores $\omega_1, \omega_2, \ldots, \omega_N$ reflecting decision-maker's preferences.

Possible approaches

- Direct scoring
- Pairwise comparison of criteria
- Pairwise comparison of actions

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- Binary irreflexive relation $P \subset \mathcal{D} \times \mathcal{D}$

- ω sought s.t. $\omega^\top \Psi(d) > \omega^\top \Psi(d') \qquad \forall d, d' \in \mathcal{D},\ dPd'$
- ω induces binary relations on $\mathcal{D}$

$$\begin{array}{llll} dP_\omega d' & \text{iff} & \omega^\top \Psi(d) > \omega^\top \Psi(d') & \text{(strict preference)} \\ dI_\omega d' & \text{iff} & \omega^\top \Psi(d) = \omega^\top \Psi(d') & \text{(indifference)} \end{array}$$

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- Hence ω_j can be seen as measure of importance of Ψ_j in P

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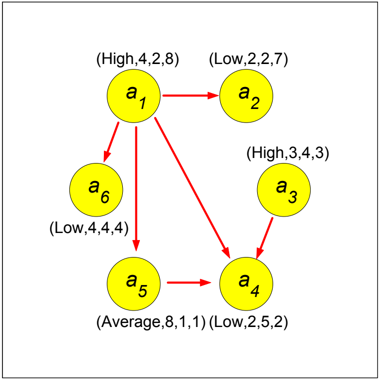
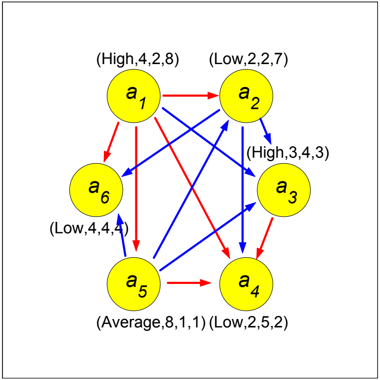

(High,4,2,8)
(Low,2,2,7)
a_1
a_2
(High,3,4,3)
a_6
a_3
(Low,4,4,4)
a_5
a_4
(Average,8,1,1)
(Low,2,5,2)


(High,4,2,8)
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d_3	High	3	4	3
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d_5	Average	8	1	1
d_6	Low	4	4	4

	Ψ_{11}	Ψ_{12}	Ψ_{13}	Ψ_2	Ψ_3	Ψ_4
d_1	1	0	0	4	2	8
d_2	0	0	1	2	2	7
d_3	1	0	0	3	4	3
d_4	0	0	1	2	5	2
d_5	0	1	0	8	1	1
d_6	0	0	1	4	4	4

$$
\begin{aligned}
\Psi_{11}(d) &= \begin{cases} 1, & \text{if } d \text{ in } C_1 \text{ takes the value "High"} \\ 0, & \text{else} \end{cases} \\
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\Psi_i(d) &= \text{score of } d \text{ according to } C_i, \qquad i = 2, \ldots, 4
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Partial information

Constraints on ω

Assume given Ω : non-empty polyhedral cone

$$\Omega = \left\{\omega : \; q_j^\top \omega \geq 0 \qquad \forall j \in J\right\}$$

$(0 \leq |J| < \infty)$

- We seek ω in Ω_P,

$$\Omega_P = \left\{\omega \in \Omega : \; \omega^\top \Psi(d) > \omega^\top \Psi(d') \qquad \forall dPd'\right\}$$

$\Omega_P \neq \emptyset$ iff $0 \notin \left(conv(\{\Psi(d) - \Psi(d') : dPd'\}) + cone(\{q_j : j \in J\})\right)$

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$(0 \leq |J| < \infty)$

- We seek ω in Ω_P,

$$\Omega_P = \left\{\omega \in \Omega : \; \omega^\top \Psi(d) > \omega^\top \Psi(d') \qquad \forall dPd'\right\}$$

$\Omega_P \neq \emptyset$ iff $0 \notin \left(conv(\{\Psi(d) - \Psi(d') : dPd'\}) + cone(\{q_j : j \in J\})\right)$

Ω

Examples

- Criteria: "the higher, the better" thus

$$\omega_j \geq 0$$

- $\omega_i \leq K_{ij}\omega_j$
- If d, d' : known to be indifferent,

$$\begin{aligned} (\Psi(d) - \Psi(d'))^\top \omega &\geq 0 \\ (\Psi(d') - \Psi(d))^\top \omega &\geq 0 \end{aligned}$$

- ...

$\Omega_P = \left\{ x \in \Omega : \; x^\top \Psi(d) > x^\top \Psi(d') \qquad \forall dPd' \right\}$

- In case $\omega \in \Omega_P$ exists, then highly desirable that $\omega^\top \Psi(d) - \omega^\top \Psi(d')$ should be not only positive but high in all pairs $d, d' \in \mathcal{D}, dPd'$
- Achieved if we maximize the lowest slack,

$$\min_{dPd'} \left\{ \omega^\top \Psi(d) - \omega^\top \Psi(d') \right\}$$

- Maximizing over Ω_P the lowest slack is, as soon as $\Omega_P \neq \emptyset$, an optimization problem with unbounded solution
- Hence, normalization condition which enables us to identify ω and $\vartheta\omega$, $\vartheta > 0$

$$\max_{\omega \in \Omega} \frac{\min_{dPd'} \left\{ \omega^\top \Psi(d) - \omega^\top \Psi(d') \right\}}{\gamma(\omega)}$$

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If

- $\Omega \subset \mathbb{R}^N_+$
- $\gamma : \ell_1$

then $\gamma(\omega) = e^\top \omega \, \forall \omega \in \Omega$, thus

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Example

$$\begin{array}{ll} \min & \gamma(\omega) \\ \text{s.t.} & \omega^\top(\Psi(d) - \Psi(d')) \geq 1 \quad \forall dPd' \\ & \omega \in \Omega_P \end{array}$$

	Ψ_{11}	Ψ_{12}	Ψ_{13}	Ψ_2	Ψ_3	Ψ_4
d_1	1	0	0	4	2	8
d_2	0	0	1	2	2	7
d_3	1	0	0	3	4	3
d_4	0	0	1	2	5	2
d_5	0	1	0	8	1	1
d_6	0	0	1	4	4	4

Example

$$
\begin{array}{ll}
\min & \gamma(\omega) \\
\text{s.t.} & \omega^\top(\Psi(d) - \Psi(d')) \geq 1 \quad \forall dPd' \\
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d_6	0	0	1	4	4	4

Constraints

- $\omega \geq 0$
- $\omega_{11} \geq \omega_{12} \geq \omega_{13}$
- $\omega_4 \leq 5\omega_3$

Example

$$\begin{array}{ll} \min & \gamma(\omega) \\ \text{s.t.} & \omega^\top(\Psi(d) - \Psi(d')) \geq 1 \quad \forall dPd' \\ & \omega \in \Omega_P \end{array}$$

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d_5	0	1	0	8	1	1
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Solution

- $(\omega_{11}, \omega_{12}, \omega_{13}, \omega_2, \omega_3, \omega_4) = (0.3418, 0.0, 0.0, 0.3397, 0.0531, 0.2654)$
- $\omega^\top \Psi(d_i) = (3.9302, 2.6435, 2.3695, 1.4757, 3.0363, 2.6329)$
- $d_1 P_\omega d_5 P_\omega d_2 P_\omega d_6 P_\omega d_3 P_\omega d_4$

$$
\begin{array}{rcl}
I_+ & := & \{(d,d') \in \mathcal{D} \times \mathcal{D} :\ dPd'\} \\
I_- & := & \{(d',d) \in \mathcal{D} \times \mathcal{D} :\ dPd'\} \\
u_{dd'} & := & \Psi(d) - \Psi(d') \in \mathbb{R}^N \qquad \forall (d,d') \in I (I := I_+ \cup I_-) \\
\Omega_P & := & \left\{\omega \in \Omega :\ \omega^\top \Psi(d) > \omega^\top \Psi(d') \qquad \forall d, d' \in \mathcal{D},\ dPd'\right\} \\
\|\cdot\| & := & \gamma^\circ
\end{array}
$$

(P)

$$
\begin{array}{ll}
\min & \|\omega\|^\circ \\
\text{s.t.} & y_i(\omega^\top u_i + \beta) \geq 1\, \forall i \in I \\
& \omega \in \Omega
\end{array}
$$

(Q)

$$
\begin{array}{ll}
\min & \gamma(\omega) \\
\text{s.t.} & y_{dd'}(\omega^\top u_{dd'} + \beta) \geq 1\, \forall (d,d') \in I \\
& \omega \in \Omega
\end{array}
$$

Th: If (ω, β) : optimal to (Q), then $\beta = 0$

$$
\begin{array}{rcl}
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$$

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$$
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\end{aligned}
$$

Th: If (ω, β) : optimal to (Q), then $\beta = 0$

Optimality conditions for (Q)

$$\begin{array}{ll} \min & \gamma(\omega) \\ \text{s.t.} & \omega^\top (\Psi(d) - \Psi(d')) \geq 1 \, \forall d, d' \in \mathcal{D}, \, dPd' \\ & \omega \in \Omega \end{array} \qquad (Q)$$

Th: Let $\omega^* \in \Omega$, feasible for (Q). Following statements are equivalent:

1. ω^* optimal for (Q)
2. $\exists \lambda = (\lambda_{dd'})_{dPd'}, \, \mu \in \mathbb{R}^J$ s.t.

$$\begin{array}{rcl} \lambda, \mu & \geq & 0 \\ \sum_{d \in \mathcal{D}} \Delta_d \Psi(d) + \sum_{j \in J} \mu_j q_j & \in & \partial\gamma(\omega^*) \\ \sum_{d' :\, dPd'} \lambda_{dd'} - \sum_{d' : d' P d} \lambda_{d'd} & = & \Delta_d \qquad \forall d \in \mathcal{D} \\ \lambda_{dd'} \left({\omega^*}^\top (\Psi(d) - \Psi(d')) - 1 \right) & = & 0 \quad \forall d, d' \in \mathcal{D}, \, dPd' \\ \mu_j \left({\omega^*}^\top q_j \right) & = & 0 \quad \forall j \in J \end{array}$$

Playing with the choice of the norm

For (Q):

Let $\omega^* \in \Omega$. Following statements are equivalent:

1. ${\omega^*}^\top \Psi(d) > {\omega^*}^\top \Psi(d') \qquad \forall d, d' \in \mathcal{D}, dPd'$
2. $\exists \gamma$, norm in $\mathbb{R}^N$, $\vartheta > 0$ s.t. $\vartheta \omega^* \in \mathcal{S}(Q)$.

Step 0: Initialize:

- Choose a norm $\gamma(\cdot)$ in $\mathbb{R}^N$.
- Take Ω_1, polyhedral cone modelling relative importance of criteria.
- Construct, by pairwise comparison among some elements of $\mathcal{D}$, P_1.
- Set $k = 1$ and go to Step 1.

Step k : Find ω^k, optimal solution to

$$
\begin{array}{ll}
\min & \gamma(\omega) \\
\text{s.t.} & \omega^\top (\Psi(d) - \Psi(d')) \geq 1 \, \forall d, d' \in \mathcal{D},\, dP_k d' \\
 & \omega \in \Omega_k,
\end{array}
$$

and find $d^k \in \mathcal{D}$, optimal to $\max_{d \in \mathcal{D}} \Psi(d)^\top \omega^k$ If (d^k, ω^k) is considered to be satisfactory then STOP with d^k as optimal solution and ω^k as vector of scores.

Else, enlarge P_k (e.g. by showing some d preferred to d^k) or reduce Ω_k. GoTo Step $k+1$.

(Toy) example: a 4-objective knapsack problem

$$\begin{array}{ll} \max & \left(r_1^\top z, \ldots, r_4^\top z\right) \\ \text{s.t.} & \omega^\top z \leq 30 \\ & z_i \in \{0,1\},\ i = 1, 2, \ldots, 10, \end{array}$$

r_1	7	6	1	9	8	2	7	7	10	4
r_2	2	1	−2	3	4	−8	2	6	9	−6
r_3	10	2	4	9	1	5	3	6	3	9
r_4	−1	4	−15	4	1	1	−6	7	−7	−1
ω	7	1	6	4	7	3	6	6	10	4

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r_4	−1	4	−15	4	1	1	−6	7	−7	−1
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High cardinality of $\mathcal{D}$ advices against the use of any methodology which starts with a complete enumeration of $\mathcal{D}$

(Toy) example: a 4-objective knapsack problem

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$$

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r_4	−1	4	−15	4	1	1	−6	7	−7	−1
ω	7	1	6	4	7	3	6	6	10	4

Sample some feasible solutions and define strict preferences among them, leading to a strict preference P, from which a scoring vector ω and an action d are obtained.

The single-objective knapsack problems

$$
\begin{array}{ll}
\max & r_i^\top z \\
\text{s.t.} & \omega^\top z \leq 30 \\
 & z_j \in \{0,1\} \quad \forall j
\end{array}
$$

Optimal solutions y_i

y_i											$r_i^\top y_i$
y_1	0	1	0	1	0	1	1	1	1	0	41
y_2	0	1	0	1	1	0	0	1	1	0	23
y_3	1	0	1	1	0	1	0	1	0	1	43
y_4	0	1	0	1	1	1	0	1	0	0	17

- DM: asked to sort the actions in $\{y_1, y_2, y_3, y_4\}$, thus defining $(\mathcal{D}, P_1) : y_2 P_1 y_4 P_1 y_1 P_1 y_3$
- Solving the SVM problem with linear constraints for $\Omega_1 = \mathbb{R}^N_+$ and P_1,

$$\begin{aligned} \omega^1 &= (0, 0.120879121, 0, 0.131868132) \\ d^1 &= (0, 1, 0, 1, 1, 0, 0, 1, 1, 0) \end{aligned}$$

- Such solution: not considered to be satisfactory, and P_1 is enriched, yielding P_2. In particular, the decision maker provides some feasible action, $y_5 = (1, 1, 0, 1, 0, 0, 0, 1, 1, 0)$, and $P_2 := P_1 \cup \{(y_5, d^1)\}$
- SVM Problem solved for P_2 :

$$\begin{aligned} \omega^2 &= (0, 0.199386503, 0.210122699, 0.246165644) \\ d^2 &= (1, 1, 0, 1, 1, 0, 0, 1, 0, 1). \end{aligned}$$

- d^2 : considered to be acceptable, and the process stops, yielding d^2 as optimal action, and ω^2 as vector of scores.

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And that's all ...

- In conclusion ...
- Thanks 4 your attention
- Questions???

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