Dantzig-Wolfe Reformulation for the Network Pricing Problem with Connected Toll Arcs

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Abstract

This work considers a pricing problem on a network with connected toll arcs and proposes a Dantzig-Wolfe reformulation for it. The model is solved with column generation and the gap between the optimal integer value and the linear relaxation optimal value is shown to be at least as good as the one from the mixed-integer formulation proposed in the literature. Numerical results show that in many cases it performs strictly better.
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Consider a highway network owned by a company, which imposes tolls on its arcs in order to maximize its revenue. Toll arcs are connected such that they constitute a single path. Users travel on the network and seek their minimum cost path. A bilevel programming framework is used to model this problem, where the leader or first level is the company and followers or second level are network users, also called commodities. This Highway Problem (HP) has been proved to be strongly $\mathcal{NP}$-hard [1]. Furthermore, Heilporn et al [2] proposed a mixed-integer formulation, and developed an efficient branch-and-cut framework.

Let $\mathcal{A}$ be the set of paths a of the highway, corresponding to pairs of entry and exit nodes, and $\mathcal{K}$ the set of commodities k, corresponding to pairs of origin and destination nodes. Each highway path has a fixed cost of c_a^k, which includes all the fixed costs commodity k incurs when using the path a: the cost for reaching the entry node from its origin, the cost for using the path and the cost for reaching the destination from the exit node. A toll free path between the origin and destination nodes of each commodity exists and its cost is denoted by c_{od}^k. Each commodity k has a demand of η^k. Variable t_a, $a \in \mathcal{A}$, represents the toll imposed by the leader on path a. Further, x_a^k, $a \in \mathcal{A}$ and $k \in \mathcal{K}$, is equal to 1 if commodity k uses toll path a and 0 otherwise, and we introduce another set of variables $p_a^k = x_a^k t_a$. The bilevel problem that we omit for lack of space (see [1]) can be reformulated as a single level non-linear optimization problem. The follower optimization problem can be separated for each commodity, and it can be replaced by constraints stating explicitly that the used path is the shortest one. The one level non linear (HP) can be written as follows [1]:

$$
\begin{aligned}
\max_{t,x} \quad & \sum_{a\in\mathcal{A}_k}\sum_{k\in\mathcal{K}} \eta^k p_a^k, && && \text{(1a)}\\
\text{s.t} \quad & \sum_{a\in\mathcal{A}_k}\Big[(c_a^k - c_{od}^k)x_a^k + p_a^k\Big] - t_b \le c_b^k - c_{od}^k && \forall k\in\mathcal{K}, \forall b\in\mathcal{A}_k, && \text{(1b)}\\
& \sum_{a\in\mathcal{A}_k}\Big[(c_a^k - c_{od}^k)x_a^k + p_a^k\Big] \le 0 && \forall k\in\mathcal{K}, && \text{(1c)}\\
& \sum_{a\in\mathcal{A}_k} x_a^k \le 1 && \forall k\in\mathcal{K}, && \text{(1d)}\\
& p_a^k = t_a x_a^k && \forall k\in\mathcal{K}, \forall a\in\mathcal{A}_k, && \text{(1e)}\\
& x_a^k \in \{0,1\} && \forall k\in\mathcal{K}, \forall a\in\mathcal{A}_k, && \text{(1f)}\\
& t_a \ge 0 && \forall a\in\mathcal{A}. && \text{(1g)}
\end{aligned}
$$

We propose a Dantzig-Wolfe reformulation for (HP). For each path $a \in \mathcal{A}$, a feasible solution is represented by the convex combination of $(x_a^k, t_a, p_a^k)^j$ values. We associate to the j^{th} solution a variable $\lambda_a^j \ge 0$. The master problem is defined by equations (1a) to (1d). We also impose $\sum_{j\in\mathcal{J}} \lambda_a^j = 1$ and $x_a^k = \sum_{j\in\mathcal{J}} \lambda_a^j x_a^{k,j} \in \{0,1\}$. Since the number of variables λ_a^j is exponential we use column generation to solve it. The pricing problem to determine a column candidate to enter the basis is separable and can thus be solved independently for each path. Further, this pricing problem, though nonlinear, is easy, i.e. it can be solved in polynomial time.

We can show that, for this Dantzig-Wolfe reformulation for (HP), the gap between the integer optimal value and the linear relaxation optimal value is equal to or smaller than the equivalent gap for the mixed-integer model present in the literature [2]. Numerical results on random generated instances show that this difference is significant, in particular when the number of toll paths is small or the number of commodities is big. For instances with 90 commodities and 20 toll paths, the Dantzig-Wolfe reformulation is able to halve the gap.

We are currently investigating different branching strategies to develop an efficient branch-and-price framework. Moreover we plan to extend this to a branch-and-cut-and-price framework, including the valid inequalities developed for this problem by [2], as they have been shown to be very efficient.

References

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