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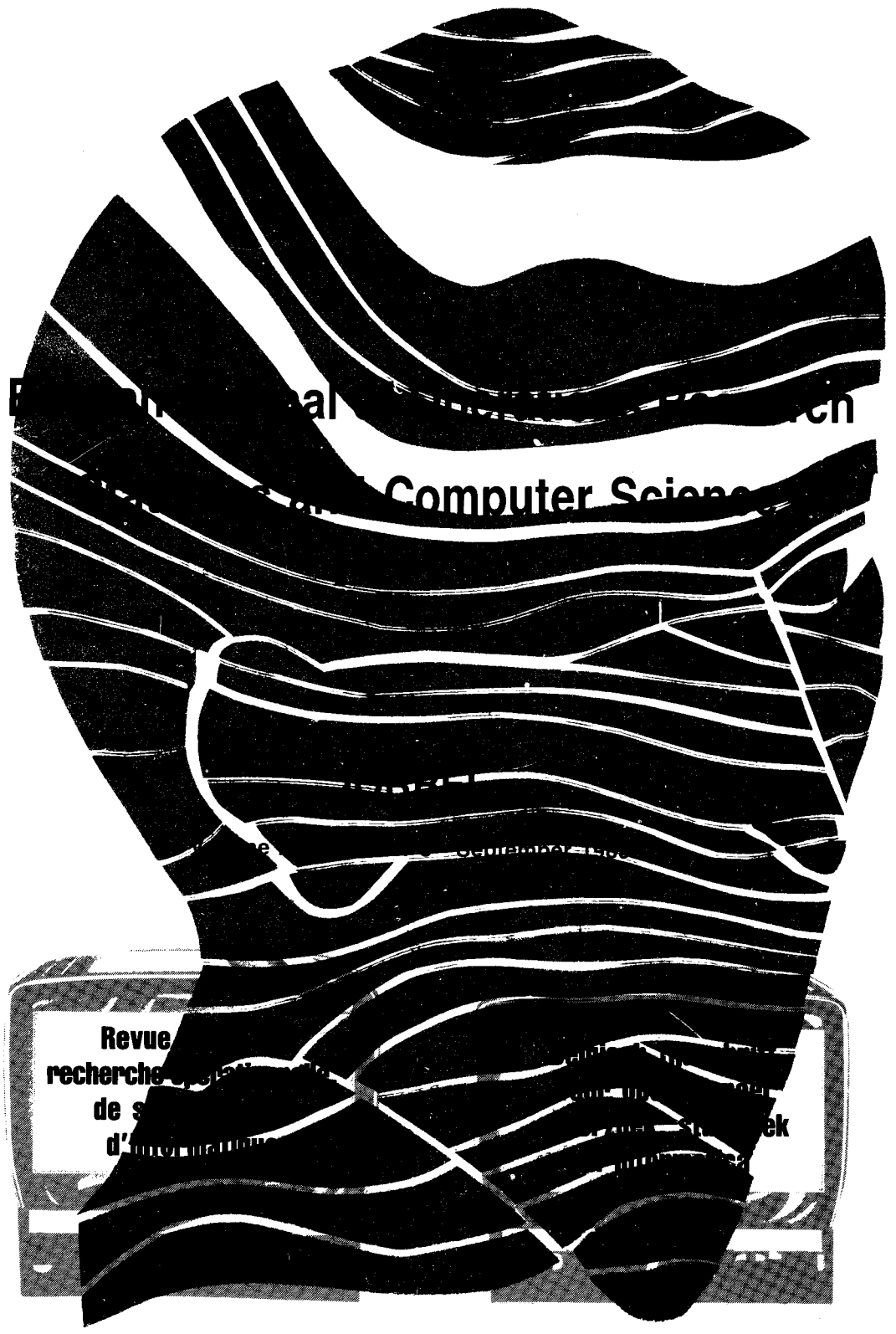

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BELGISCH TIJDSCHRIFT VOOR OPERATIONEEL ONDERZOEK, STATISTIEK EN INFORMATICA

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LINEAR FRACTIONAL PROGRAMMING A NEW BICRITERIA APPROACH

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ABSTRACT

The linear fractional programming problem is formulated and a new algorithm for solving this type of problem is presented. In each iteration of the method one proceeds to a very narrow evaluation of the error incurred by the current approximation to the optimum. This error may be obtained by some parametric methods, but in our algorithm we use a distinct analysis based on a suggestive graphical bicriteria representation. Some computational experiments comparing the algorithms were carried out.

1. Introduction

-The aim of this work is to introduce a new algorithm to solve linear fractional programming problems by using a bicriteria linear programming approach.

-The linear fractional programming problems usually admit the following formulation:

$$\min \{f(x)=f_1(x)/f_2(x): x\in X\} \quad (P_1)$$

where:

$f_i(x)=c_i^t x+\beta_i$, $c_i\in\mathbb{R}^n$, $\beta_i\in\mathbb{R}$ for $i=1,2$

$X=\{x\in\mathbb{R}^n: Ax=b, x\geq 0\}$

$A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$ and $m,n\in\mathbb{N}$ with $m<n$

$X\neq\emptyset$ and compact

-Let us also suppose that $f_1(x)\geq 0$ and $f_2(x)>0 \quad \forall x\in X$.

-It should be noted that in the published articles on linear fractional programming, in addition to the formulation given for (P_1), only the condition $f_2(x)>0 \ \forall x\in X$ is required instead of the conditions here imposed: $f_1(x)\geq 0 \ \forall x\in X$ and $f_2(x)>0 \ \forall x\in X$.

-However, in practice, if $\exists x'\in X$: $f_1(x')<0$ the problem in question can be easily transformed into another equivalent problem, in the sense that it admits the same set of optimum solutions and observes the general given formulation.

-In Climaco et al [7] and Mykowski et al [6], it is suggested the transformation of (P_1) into a bicriteria problem:

$$\min_{x\in X} f_1(x) \ ; \ \max_{x\in X} f_2(x)$$

or it be: (P_2)

$$\min_{x\in X} F(x), \text{ where } F(x)=\begin{bmatrix} f_1(x) \\ -f_2(x) \end{bmatrix}$$

-With this formulation, we intend to fix the set of non dominated solutions of this problem. The meaning of non dominated solution is the following :

-Let $x_1,x_2\in X$, we say that x_1 dominates x_2 ($x_1 D x_2$) iff $F(x_1)\leq F(x_2)$, where:

$F(x_1)\leq F(x_2)$ iff $f_1(x_1)\leq f_1(x_2)$, $-f_2(x_1)\leq -f_2(x_2)$ and $F(x_1)\neq F(x_2)$.

-Identifying by X_D the set of dominated solutions, i. e,

$X_D \equiv \{x \in X: \exists x' \in X,\ x' D x\}$, the set of not dominated solutions, identified by X_N, is given by $X_N \equiv X \setminus X_D$.

-Throughout this paper we will designate the set of optimum solutions of (P_1) by X^*, that is, $X^* \equiv \{x^* \in X: f(x^*) \leq f(x)\ \forall x \in X\}$.

Theorem 1 [1]

If for the problem (P_1) $f_1(x) > 0\ \forall x \in X$, then $X_N \supseteq X^*$.

Theorem 2

There is at least one point $x^* \in X^*$ which is an extreme point of X.

-Based on theorems 1 and 2, it is easily acknowledged that from the solution of linear programs of the type:

$$\min\{h_\alpha(x) = f_1(x) - \alpha f_2(x): x \in X\} \qquad (P_\alpha)$$

where the parameter α takes suitable values, we end by getting the optimum solution to (P_1). We will identify by $\hat{x}_\alpha$ an optimal solution (extreme point) of the program (P_α) and by $\hat{h}_\alpha$ the respective optimal value $(\hat{h}_\alpha = h_\alpha(\hat{x}_\alpha))$.

-As in the presented method, the foundation of methods -such as the one of Martos [1] (in which the objective function (o.f.) changes from simplex iteration to simplex iteration), the one of Isbell and Marlow [3], the one of Frankel, Novaes and Pollack [4] [2] and the methods described in [13] and [14]- lies upon the solution of linear programs (P_α), with the real parameter α varying in a suitable range of values.

-The methods described in [13] and [14] are dedicated to nonlinear programming and are based on the application of the classic bisection-method to the determination of the root of the equation, $\eta(\alpha) = 0$ with $\eta(\alpha) = \hat{h}_\alpha$, where η is a concave picewise linear function.

-Other approaches, either have other essential aims, different from the solution of (P_1), as it is the case for the method introduced by Wolf [12] (more concerned with the analysis of this programs than with solving the problem) or are based on the classic method of Charnes and Cooper [2]. In any case they

[1] Taking into account the formulation, given to (P_1) and (P_2) we remark that:

If $\exists x' \in X: f_1(x') = 0$, then $x' \in X^*$, however may $\exists x' \in X: f_1(x') = 0$ and $x' \notin X_N$.

[2] See the version of this method presented by Nykowski and Zolkiewski [6] that solve the linear fractional programming problem using an approach like (P_2).

present certain computational disadvantages comparatively to the ones already indicated.

-The bicriteria approach method although could be considered in some aspects similar to some of the referred methods, mainly the ones based on the bisection-method (because, like them, it is possible to calculate a rather narrow value of the error incurred by the approximation to the optimum given by the point obtained in each iteration) is quite different from all of them in the way how the parameter α is obtained. The fixing of this parameter during the iterative process, is always made from two suboptimal solutions with values of the numerator and denominator functions of the o.f., higher than and lower than the ones obtained for the optimum solution to (P_1), respectively. The development of the method enables the images of these points in the objectives space of (P_2) to become more and more close, but continueing to frame the image or images of the optimum solution or solutions of (P_1) in this space.

2. The bicriteria approach method

-The method we are going to present, can be considered related to "NISE" [3] studied by Cohon [8], with respect to the way it determines the error incurred when it approximates the optimum by the iterated point.

-Let $G:\mathbb{R}^n \dashrightarrow \mathbb{R}^2$ such that $G(x)=(f_2(x),f_1(x))$, the application of the method leads to a sucession of special bidimensional bisections in order to determine a gradually more "tighter" region to which the image by "G", of the searched optimum, belongs.

-Before presenting the new algorithm it is convenient to introduce some results in order to facilitate the proof of its convergence.

-The theorems 3 and 4, to be presented, set up the fundamental results of this method and they will enable to make the proof of the convergence (enunciated in theorem 5).

-With the aim of simplifying the theorems proofs several lemmas are presented in appendix; 5.1, 5.2 and 5.4 have the objective of establishing the criteria correction for stopping the algorithm and 5.3 is used to avoid the redundancy in the confirmation of certain conditions.

-From now on (P_α) will only be defined for positive α $(\alpha>0)$.

---

[3] An interactive multiobjective linear programming method.

Lemma 3.1

$\forall \hat{x}_\alpha$ solution of (P_α), and $\forall x^* \in X^*$ the following system of inequalities is valid:

$$f_1(x^*)-\alpha f_2(x^*) \geq h_\alpha(\hat{x}_\alpha)$$
$$f_1(x^*)-f(\hat{x}_\alpha) f_2(x^*) \leq 0$$

The proof is trivial.

□

Theorem 3

$\forall \hat{x}_\alpha$, solution of (P_α), $\forall x^* \in X^*$ we have:

$$(a)\ h_\alpha(\hat{x}_\alpha)<0 \Rightarrow f_1(x^*) \leq f_1(\hat{x}_\alpha) \wedge f_2(x^*) \leq f_2(\hat{x}_\alpha)$$
$$(b)\ h_\alpha(\hat{x}_\alpha)>0 \Rightarrow f_1(x^*) \geq f_1(\hat{x}_\alpha) \wedge f_2(x^*) \geq f_2(\hat{x}_\alpha)$$
$$(c)\ h_\alpha(\hat{x}_\alpha)=0 \Rightarrow f(x^*)=f(\hat{x}_\alpha)$$

proof:

(a) $h_\alpha(\hat{x}_\alpha)<0 \Leftrightarrow \alpha-f(\hat{x}_\alpha)>0$

From lemma 3.1, we have:

$$-f_1(x^*)+\alpha f_2(x^*) \leq f_2(\hat{x}_\alpha)(\alpha-f(\hat{x}_\alpha))$$
$$f_1(x^*)-f(\hat{x}_\alpha) f_2(x^*) \leq 0$$
$$---$$
$$(\alpha-f(\hat{x}_\alpha)) f_2(x^*) \leq (\alpha-f(\hat{x}_\alpha)) f_2(\hat{x}_\alpha)$$

and since by hypothesis $\alpha-f(x^i)>0$ we may conclude that $f_2(x^*) \leq f_2(\hat{x}_\alpha)$. From this result and from the 2nd inequality in the above system, we may conclude: $f_1(x^*) \leq f_1(\hat{x}_\alpha)$.

□

(b) proof as in (a).

(c) proof is trivial.

Theorem 4

Let (P_{α_A}) and (P_{α_B}) such that $\hat{h}_{\alpha_A}<0$, $\hat{h}_{\alpha_B}>0$, $f_2(\hat{x}_{\alpha_A}) \neq f_2(\hat{x}_{\alpha_B})$ and $(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B})/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})) \notin \{\alpha_A, \alpha_B\}$.

Let $\bar{\alpha}=\min\{(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})), f(x_A), f(x_B)\}$.

Then $f_1(\hat{x}_{\alpha_B}) \leq f_1(\hat{x}_{\bar{\alpha}}) \leq f_1(\hat{x}_{\alpha_A})$ and $f_2(\hat{x}_{\alpha_B}) \leq f_2(\hat{x}_{\bar{\alpha}}) \leq f_2(\hat{x}_{\alpha_A})$ [4].

proof:

[4]Have in attention that $\hat{x}_{\bar{\alpha}}$ is the optimal solution of $(P_{\bar{\alpha}})$.

-By the hypothesis $f_1(\hat{x}_{\bar{\alpha}})-f_1(x)\leqq\bar{\alpha}(f_2(\hat{x}_{\bar{\alpha}})-f_2(x))\ \forall x\in X$ (*).

-Since[5] $f_1(\hat{x}_{\bar{\alpha}})-f_1(\hat{x}_{\alpha_B})\geqq 0$, $f_2(\hat{x}_{\bar{\alpha}})-f_2(\hat{x}_{\alpha_B})\leqq 0$, $\bar{\alpha}>0$ [6] and the particular case of the relation (*), for $x=\hat{x}_{\alpha_B}$, we get:

$$f_1(\hat{x}_{\bar{\alpha}})-f_1(\hat{x}_{\alpha_B})\leqq\bar{\alpha}(f_2(\hat{x}_{\bar{\alpha}})-f_2(\hat{x}_{\alpha_B})),$$

-We can conclude that $f_2(\hat{x}_{\bar{\alpha}})-f_2(\hat{x}_{\alpha_B})\geqq 0$. Also the relation (*) for the particular case of $x=\hat{x}_{\alpha_A}$, becomes:

$$f_1(\hat{x}_{\bar{\alpha}})-f_1(\hat{x}_{\alpha_A})\leqq\bar{\alpha}(f_2(\hat{x}_{\bar{\alpha}})-f_2(\hat{x}_{\alpha_A})),$$

and so its easily concluded that $f_1(\hat{x}_{\bar{\alpha}})-f_1(\hat{x}_{\alpha_A})\leqq 0$, thus completing the proof.

□

-Since we already have the necessary results for justifying the bicriteria approach method, this will now be described by the corresponding algorithm, where ε identifies the upper bound for the error incurred when the point obtained in the current iteration is taken as an approximation to the optimum, and x^* identifies this approximation determined for ε not greater than the admissible error.

-If the value of ε associated with the approximation is zero, this means that the current point belongs to X^* and so it is an optimal solution to the problem (P_1).

Algorithm:

0. Choose a real positive value for α^0 and choose a real value not less than 1 for k;
   Determine $\hat{x}_{\alpha^0}$;
   If $\hat{h}_{\alpha^0}=0$ then begin $x^*\leftarrow\hat{x}_{\alpha^0}$; $\varepsilon\leftarrow 0$; STOP end;
   If $\hat{h}_{\alpha^0}<0$ then begin
   $\alpha_A\leftarrow\alpha^0$; $\hat{x}_{\alpha_A}\leftarrow\hat{x}_{\alpha^0}$; $\hat{x}_{\alpha}\leftarrow\hat{x}_{\alpha^0}$; repeat
   $\alpha\leftarrow f(\hat{x}_{\alpha})/k$;
   Determine $\hat{x}_{\alpha}$;
   If $\hat{h}_{\alpha}=0$ then begin
   $x^*\leftarrow\hat{x}_{\alpha}$; $\varepsilon\leftarrow 0$; STOP
   end
   until $h_{\alpha}(x_{\alpha})\geqq 0$;
   $\alpha_B\leftarrow\alpha$; $\hat{x}_{\alpha_B}\leftarrow\hat{x}_{\alpha}$
   end;

[5] By the lemma 4.3 in appendix.

[6] In according whith the lemma 4.2 in apendix, having in attention that $\alpha_B>0$.

If $\hat{h}_{\alpha^0} > 0$ then begin

$\alpha_B \leftarrow \alpha^0$; $\hat{x}_{\alpha_B} \leftarrow \hat{x}_{\alpha^0}$; $\alpha_A \leftarrow f(\hat{x}_{\alpha^0})$;

Determine $\hat{x}_{\alpha_A}$;

If $\hat{h}_{\alpha_A} = 0$ then begin

$x^* \leftarrow \hat{x}_{\alpha_A}$; $\varepsilon \leftarrow 0$; STOP

end

end;

If $f_2(\hat{x}_{\alpha_A}) = f_2(\hat{x}_{\alpha_B})$ then begin

$x^* \leftarrow \hat{x}_{\alpha_B}$; $\varepsilon \leftarrow 0$; STOP

end;

$i \leftarrow 0$; go to step 1;

1. If $(f_1(\hat{x}_{\alpha_A}) - f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A}) - f_2(\hat{x}_{\alpha_B})) \notin \{\alpha_A, \alpha_B\}$ then

begin

$x^* \leftarrow x'$ such that $x' \in \{\arg\min\{f(\hat{x}_{\alpha_A}), f(\hat{x}_{\alpha_B})\}\}$;

$\varepsilon \leftarrow 0$;

STOP

end;

$i \leftarrow i+1$; $\alpha^i \leftarrow \min\{(f_1(\hat{x}_{\alpha_A}) - f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A}) - f_2(\hat{x}_{\alpha_B})), f(\hat{x}_{\alpha_A}), f(\hat{x}_{\alpha_B})\}$;

Determine $\hat{x}_{\alpha^i}$;

If $\hat{h}_{\alpha^i} = 0$ then begin $x^* \leftarrow \hat{x}_{\alpha^i}$; $\varepsilon \leftarrow 0$; STOP end;

If $\hat{h}_{\alpha^i} = h_{\alpha^i}(\hat{x}_{\alpha_A})$ then begin $x^* \leftarrow \hat{x}_{\alpha_A}$; $\varepsilon \leftarrow 0$; STOP end;

If $\hat{h}_{\alpha^i} < 0$ then begin $\hat{x}_{\alpha_A} \leftarrow \hat{x}_{\alpha^i}$; $\alpha_A \leftarrow \alpha^i$; go to step 2 end;

If $\hat{h}_{\alpha^i} > 0$ then begin $\hat{x}_{\alpha_B} \leftarrow \hat{x}_{\alpha^i}$; $\alpha_B \leftarrow \alpha^i$; go to step 2 end;

2. If $f_2(\hat{x}_{\alpha_A}) = f_2(\hat{x}_{\alpha_B})$ then begin $x^* \leftarrow \hat{x}_{\alpha_B}$; $\varepsilon \leftarrow 0$; STOP end;

$\varepsilon \leftarrow |f_1(\hat{x}_{\alpha_B})/f_2(\hat{x}_{\alpha_A}) - \min\{f(\hat{x}_{\alpha_A}), f(\hat{x}_{\alpha_B})\}|$;

If $\varepsilon >$ admissible error then begin

go back to step 1

end

else begin

$x^* \leftarrow x'$ such that $x' \in \{\arg\min\{f(\hat{x}_{\alpha_A}), f(\hat{x}_{\alpha_B})\}\}$; STOP

end

end(of algorithm).

Theorem 5

$\forall \alpha^0 \in \mathbb{R}^+$, $\forall k \in [1, +\infty[$, the introduced algorithm determines the solution to (P_1) after a finite number of steps.

proof:

Application of STEP 0;

-In step 0, if $\hat{h}_{\alpha^0}=0$ then by the theorem 3 (c) $\hat{x}_{\alpha^0}\in X^*$. Otherwise $\hat{h}_{\alpha^0}>0$ or $\hat{h}_{\alpha^0}<0$.

(i) Suppose that $\hat{h}_{\alpha^0}<0$.

-According to the algorithm it results that $\alpha_A=\alpha^0$, $\hat{x}_{\alpha_A}=\hat{x}_{\alpha^0}$, $\alpha=f(\hat{x}_{\alpha^0})/k$ and thence we determine $\hat{x}_\alpha$. If $\hat{h}_\alpha<0$ then we do $\alpha=f(\hat{x}_\alpha)/k$ and again we determine $\hat{x}_\alpha$. This procedure will be repeated until $\hat{h}_\alpha\geq 0$.

> -It should be noted that $\hat{h}_\alpha<0 \Rightarrow f(\hat{x}_\alpha)<f(\hat{x}_{\alpha'})$,[7] $\hat{x}_{\alpha'}$ being an extreme point calculated by the method immediately before $\hat{x}_\alpha$ .Whenever $\hat{h}_\alpha<0$, this means that the extreme point $\hat{x}_\alpha$ decrease the objective function with respect to $\hat{x}_{\alpha'}$. Therefore the fact that the number of extreme points of X is finite, after the execution of a finite number of these procedures, entails that we have $\hat{h}_\alpha\geq 0$.

-If $\hat{h}_\alpha=0$ then, once again by theorem 3 (c) it results that $\hat{x}_\alpha\in X^*$, otherwise, according to the algorithm, $\alpha_B=\alpha$ and $\hat{x}_{\alpha_B}=\hat{x}_\alpha$.

(ii) Suppose that $\hat{h}_{\alpha^0}>0$.

-According to the algorithm we get $\alpha_B=\alpha^0$, $\hat{x}_{\alpha_B}=\hat{x}_{\alpha^0}$, $\alpha_A=f(\hat{x}_{\alpha^0})$ and thence we determine $\hat{x}_{\alpha_A}$.

-$\hat{h}_{\alpha_A}\leq 0$, since $h_{\alpha_A}(\hat{x}_{\alpha^0})=0$ and $\hat{h}_{\alpha_A}\leq h_{\alpha_A}(x)\ \forall x\in X$.

-If $\hat{h}_{\alpha_A}=0$ then like before, by theorem 3 (c) it results that $\hat{x}_{\alpha_A}\in X^*$.

-Nevertheless if the optimum is not found neither by (i) nor by (ii), we'll have, in any case, $\hat{h}_{\alpha_A}<0$ and $\hat{h}_{\alpha_B}>0$ and by theorem 3:

$$f_1(\hat{x}_{\alpha_B})\leq f_1(x^*)\leq f_1(\hat{x}_{\alpha_A}) \ \wedge \ f_2(\hat{x}_{\alpha_B})\leq f_2(x^*)\leq f_2(\hat{x}_{\alpha_A})$$

-If $f_2(\hat{x}_{\alpha_A})=f_2(\hat{x}_{\alpha_B})$ [8] then $\hat{x}_{\alpha_B}\in X^*$, otherwise go to step 1.

---

[7] Have in attention that $k\geq 1$.

[8] By lemma 5.2 in appendix.

Application of STEPS 1 and 2:

-If $(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B}))=\{\alpha_A,\alpha_B\}$ we may conclude that $\{\hat{x}_{\alpha_A},\hat{x}_{\alpha_B}\}\cap X^*=\emptyset$, so $X^*\supseteq\{\arg\min\{f(\hat{x}_{\alpha_A}),f(\hat{x}_{\alpha_B})\}\}$ [9].

-Let us suppose that

$$(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B}))=\{\alpha_A,\alpha_B\}\ (*),$$

from the algorithm we have:

$$\alpha^i=\min\{(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})),f(\hat{x}_{\alpha_A}),f(\hat{x}_{\alpha_B})\}$$

and thence we determine $\hat{x}_{\alpha}i$.

-If $\hat{h}_{\alpha}i=0$ then from theorem 3 (c) $\hat{x}_{\alpha}i=X^*$, otherwise if $\hat{h}_{\alpha}i=h_{\alpha}i(\hat{x}_{\alpha_A})$ then $\hat{x}_{\alpha_A}=X^*$ [10].

-Let us suppose that the condition (*) holds, $\hat{h}_{\alpha}i=0$ and also $\hat{h}_{\alpha}i=h_{\alpha}i(\hat{x}_{\alpha_A})$.

-By theorem 4 we have:

$$f_1(\hat{x}_{\alpha_B})\leq f_1(\hat{x}_{\alpha}i)\leq f_1(\hat{x}_{\alpha_A})\ \wedge\ f_2(\hat{x}_{\alpha_B})\leq f_2(\hat{x}_{\alpha}i)\leq f_2(\hat{x}_{\alpha_A})\ (**)$$

-If $\hat{h}_{\alpha}i<0$ by theorem 3, we conclude that $f_1(x^*)\leq f_1(\hat{x}_{\alpha}i)$ and $f_2(x^*)\leq f_2(\hat{x}_{\alpha}i)$ so:

$$f_1(\hat{x}_{\alpha_B})\leq f_1(x^*)\leq f_1(\hat{x}_{\alpha}i)\ \wedge\ f_2(\hat{x}_{\alpha_B})\leq f_2(x^*)\leq f_2(\hat{x}_{\alpha}i)$$

and since $(f_1(\hat{x}_{\alpha}i),f_2(\hat{x}_{\alpha}i))=(f_1(\hat{x}_{\alpha_A}),f_2(\hat{x}_{\alpha_A}))$ (notice that by hypothesis $\hat{h}_{\alpha}i=h_{\alpha}i(\hat{x}_{\alpha_A})$), we'll have from (**) that either $f_1(\hat{x}_{\alpha}i)<f_1(\hat{x}_{\alpha_A})$ or $f_2(\hat{x}_{\alpha}i)<f_2(\hat{x}_{\alpha_A})$.

-If $\hat{h}_{\alpha}i>0$ from theorem 3 it is concluded that $f_1(\hat{x}_{\alpha}i)\leq f_1(x^*)$ and $f_2(\hat{x}_{\alpha}i)\leq f_2(x^*)$. Consequently:

$$f_1(\hat{x}_{\alpha}i)\leq f_1(x^*)\leq f_1(\hat{x}_{\alpha_A})\ \wedge\ f_2(\hat{x}_{\alpha}i)\leq f_2(x^*)\leq f_2(\hat{x}_{\alpha_A})$$

and like before from (**) we conclude that $f_1(\hat{x}_{\alpha_B})<f_1(\hat{x}_{\alpha}i)$ or $f_2(\hat{x}_{\alpha_B})<f_2(\hat{x}_{\alpha}i)$. It must be remarked that since the hypothesis of the lemma 5.3, in appendix, are hold, $\hat{h}_{\alpha}i=h_{\alpha}i(\hat{x}_{\alpha_A})$ is equivalent to $\hat{h}_{\alpha}i=h_{\alpha}i(\hat{x}_{\alpha_B})$ and therefore $(f_1(\hat{x}_{\alpha}i),f_2(\hat{x}_{\alpha}i))=(f_1(\hat{x}_{\alpha_A}),f_2(\hat{x}_{\alpha_A}))$.

-As X is convex compact with a finite number of extreme points, if we consider that $f_2(\hat{x}_{\alpha_B})=f_2(\hat{x}_{\alpha_A})$, $\hat{h}_{\alpha}i=h_{\alpha}i(\hat{x}_{\alpha_A})$ and that (*) holds in all the iterations of the method in the course

[9] See lemma 5.1 in appendix.

[10] According to lemma 5.4 in appendix.

of STEP 1, then it will only be possible to get a certain number of times either $\hat{h}_{\alpha^i}<0$ or $\hat{h}_{\alpha^i}>0$. Then, for some i, we'll necessarily have $\hat{h}_{\alpha^i}=0$. Therefore from theorem 3 (c) $\hat{x}_{\alpha^i}\in X^*$.

-Finally if in any iteration, in the course of STEP 2, it comes about that $f_2(\hat{x}_{\alpha_B})=f_2(\hat{x}_{\alpha_A})$ then $\hat{x}_{\alpha_B}\in X^*$ [11].

□

-It should be noted that values of the parameters α^o in $\mathbb{R}^+$ and k in $[1,+\infty[$, have no influence in the convergence of the method, in the sense that this will necessarily ocurr.

-A carefull analysis of the algorithm, leads to the conclusion that the more close α^o is to the optimum value the more rapid is the convergence of the algorithm. So the knowledge of an approximation to the optimum, (if it exists) must be taken into account in the choice of the value for α^o.

-Sometimes the characteristics of the o.f. of the problem can give some information about the limits of the interval to which the optimum value belongs. For instance if the coefficients of the denominator function and its independent term are positives, we can conclude the following:

Let $\frac{c_{1_p}}{c_{2_p}} = \min\{\frac{c_{1_i}}{c_{2_i}},\ i=1,\_,n\}$ and $\frac{c_{1_q}}{c_{2_q}} = \max\{\frac{c_{1_j}}{c_{2_j}},\ j=1,\_,n\}$

If $x^*\in X^*$, then:

$$\min\{\frac{c_{1_p}}{c_{2_p}},\frac{\beta_1}{\beta_2}\}\leq f(x^*)\leq\max\{\frac{c_{1_q}}{c_{2_q}},\frac{\beta_1}{\beta_2}\} \qquad (***)$$

(see formulation given to P_1).

-This conclusion derives from the Cauchy inequalities [9]:

-If $n_1,n_2,\_,n_t$ are real numbers and $d_1,d_2,\_,d_t$ are positive, then:

$$\min\{\frac{n_i}{d_i},\ i=1,\_,t\}\leq\frac{n_1+n_2+\_+n_t}{d_1+d_2+\_+d_t}\leq\max\{\frac{n_i}{d_i},\ i=1,\_t\}$$

-Without loss of generality, suppose that $0\notin\{x^*_1,\_,x^*_{t-1}\}$, with $t-1\leq n$, where x^*_i, for $i=1,\_,t-1$ identifies each of the non null components of the optimum solution x^*.

[11] From Lemma 5.2

-With $n_i=c_{1_i}x^*_i$ and $d_i=c_{2_i}x^*_i$ for $i=1,2,..,t-1$, $n_t=\beta_1$ and $d_t=\beta_2$ we have $c_{1_i}x^*_i/c_{2_i}x^*_i=c_{1_i}/c_{2_i}$ for $i=1,..,t-1$, and in these conditions we obtain the inequalities $(''')$.

-As for parameter k, we think, from computational experiments, that k=2 is a good choice.

-If α^0 takes a value not less than the optimum value of (P_1) and k takes the value 1, the method doesn't leave STEP 0 without finding the optimal solution and behaves similarly to the Isbell et al's method.

3. Evaluation of the error incurred when we take point x^k, obtained at the "k^{th}" iteration of the method, as an approximation to the solution of (P_1).

-This method allows the estimation, for each point x^i (obtained at the i^{th} iteration of the algorithm), of an upper bound for the error incurred when that point is taken as an approximation to the optimum, which is lower the higher is the order "i" of the iteration.

-This possibility is of great interest in situations where the number of extreme points in the neighbourhood of the optimum is high. In this situation not only the absolute precision $\varepsilon=0$ (in the present algorithm the upper bound for the error is designated as ε), may entail a large number iterations that are not justifiable but also in extreme cases this may have no sense, taking into account numerical considerations.

-According to the previous study we know that in each iteration of the method, once $\hat{x}_{\alpha_A}$ and $\hat{x}_{\alpha_B}$ have been determined, if $x^*\in X^*$, then:

$$f_1(\hat{x}_{\alpha_B})\leq f_1(x^*)\leq f_1(\hat{x}_{\alpha_A}) \quad \wedge \quad f_2(\hat{x}_{\alpha_B})\leq f_2(x^*)\leq f_2(\hat{x}_{\alpha_A})$$

and in these conditions:

$$f_1(\hat{x}_{\alpha_B})/f_2(\hat{x}_{\alpha_A})\leq f(x^*)\leq \min\{f(\hat{x}_{\alpha_A}),f(\hat{x}_{\alpha_B})\}.$$

-When approximating x^* by the $\arg\min\{f(\hat{x}_{\alpha_A}),f(\hat{x}_{\alpha_B})\}$, the incurred error is not greater than

$$\varepsilon=|f_1(\hat{x}_{\alpha_B})/f_2(\hat{x}_{\alpha_A})-\min\{f(\hat{x}_{\alpha_A}),f(\hat{x}_{\alpha_B})\}|.$$

-This upper bound may still be reduced by taking as lower bound for the objective function's value in X, the value of that function corresponding to the point of $G(\mathbb{R}^n)$ [12] which is

[12] With $G(x)=(f_2(x),f_1(x))$.

the intersection of the support lines of that region that go through points $(f_2(\hat{x}_{\alpha_A}),f_1(\hat{x}_{\alpha_A}))$ and $(f_2(\hat{x}_{\alpha_B}),f_1(\hat{x}_{\alpha_B}))$ respectively. That is $f_1(\bar{x})/f_2(\bar{x})$ is such that $(f_2(\bar{x}),f_1(\bar{x}))$ is a solution to the system of equations:

$$f_1(x)-\alpha_A f_2(x)=\hat{h}_{\alpha_A}$$

$$f_1(x)-\alpha_B f_2(x)=\hat{h}_{\alpha_B}$$

-It should be noted that, according to the proposed method, if $\hat{x}_{\alpha_A}$ and $\hat{x}_{\alpha_B}$ are points different from the searched optimum we have:

$$\hat{h}_{\alpha_A}<0 \iff f(\hat{x}_{\alpha_A})<\alpha_A$$

$$\hat{h}_{\alpha_B}>0 \iff f(\hat{x}_{\alpha_B})>\alpha_B$$

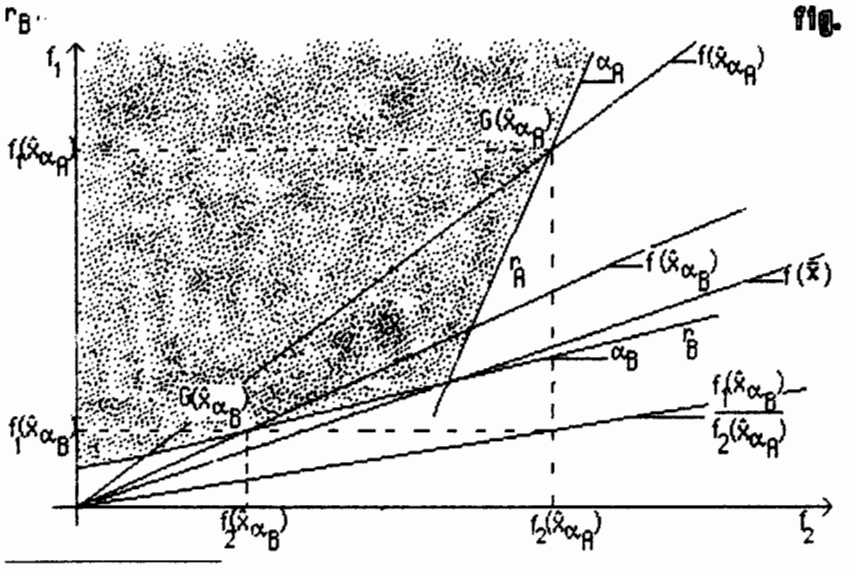
-Let us consider the figure 1. Taking into account that the lines r_A, r_B defined by the equations $h_{\alpha_A}(x)=\hat{h}_{\alpha_A}$ and $h_{\alpha_B}(x)=\hat{h}_{\alpha_B}$ respectively, are the support lines of G(X), the whole feasible region [13], is contained in the shadowed area, since:

$$h_{\alpha_A}(x)\geq\hat{h}_{\alpha_A} \quad \forall x\in X$$

$$h_{\alpha_B}(x)\geq\hat{h}_{\alpha_B} \quad \forall x\in X$$

-Therefore, taking into account that $\alpha_A,\alpha_B\in\mathbb{R}^+$ and that $f(\hat{x}_{\alpha_A})<\alpha_A$, $f(\hat{x}_{\alpha_B})>\alpha_B$, any line which passes through the origin and a point $(f_2(x),f_1(x))$ of the shadowed region, has a slope f(x) not less than the slope of the line that passes through the origin and the point $(f_2(\bar{x}),f_1(\bar{x}))$, intersection of r_A and r_B.

fig. 1



[13] The feasible region is defined by "G(X)".

-Consequently we have $f(x)\geq f(\bar{x})$ $\forall x\in X$ and in particular $f(x^*)\geq f(\bar{x})$.

-On the other hand the condition $f(\bar{x})\geq f_1(\hat{x}_{\alpha_B})/f_2(\hat{x}_{\alpha_A})$, is also verified and hence the upper bound for the error cannot be greater than the one initially considered if we take the value:

$$\delta=|f(\bar{x})-\min\{f(\hat{x}_{\alpha_A}),f(\hat{x}_{\alpha_B})\}| \qquad (1)$$

-This value "δ" chosen for upper bound of the error is considerably more narrow than the value identified by "ε". Therefore the value δ should be the one to be used as an upper bound for the error, having in mind to obtain the highest possible precision.

4. Example of Application

-We'll apply the new algorithm to the following example taking as maximum admissible error 5×10^{-3} (that is δ must not be greater than this value), $\alpha^0=0.5$ and $k=2$:

Determine $\arg\min\{f_1(x)/f_2(x): x\in X\}$

with $f_1(x)=-x_1+x_2+40$; $f_2(x)=x_1+3x_2-18$;

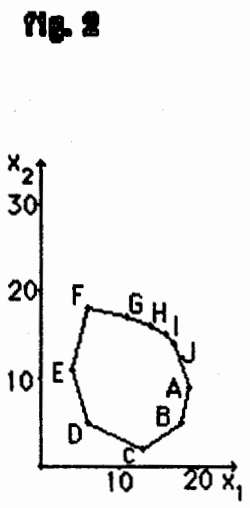
X being a set of $\mathbb{R}^2$, defined by the following constraints:

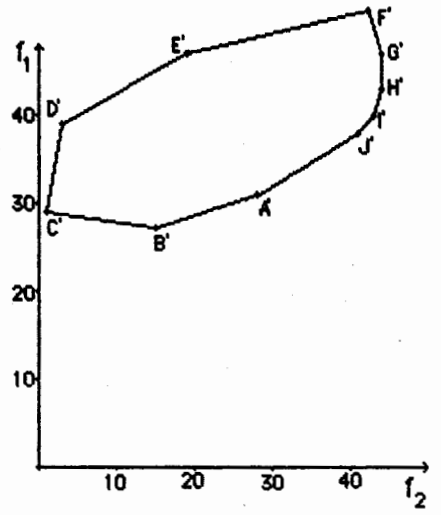
$x_1+5x_2\leq 96$; $x_1+3x_2\leq 62$; $x_1+2x_2\leq 46$;
$x_1+x_2\leq 31$; $5x_1+2x_2\leq 113$; $4x_1-x_2\leq 67$;
$3x_1-5x_2\leq 29$; $3x_1+7x_2\geq 53$; $3x_1+x_2\geq 23$;
$7x_1-2x_2\geq 6$; $x_1,x_2\geq 0$

-The decision space for this problem and the respective image, given by $f_1(x)$ and $f_2(x)$ are represented in figure 2.

-The images of A, B, C, D, E, F, G, H, I, J (extreme points of the decision space) by $f_1(x)$ and $f_2(x)$ are identified as A', B', C', D', E', F', G', H', I', J' respectively. The value of the points coordinates $P=(x_1,x_2)$ and $P'=(f_2(P),f_1(P))$ for $P=A,B,\ldots,J$, are indicated in the next table.

Fig. 2





A	B	C	D	E	F	G	H	I	J
(19,9)	(18,5)	(13,2)	(6,5)	(4,11)	(6,18)	(11,17)	(14,16)	(16,15)	(17,14)
A'	B'	C'	D'	E'	F'	G'	H'	I'	J'
(28,30)	(15,27)	(1,29)	(3,39)	(19,47)	(42,52)	(44,46)	(44,42)	(43,39)	(41,37)

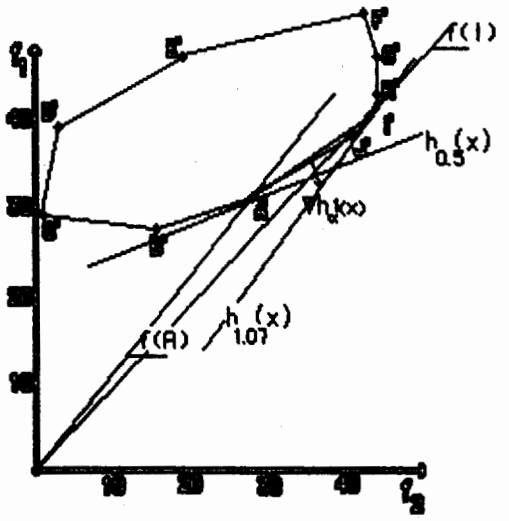
-By applying "STEP 0" we determine $\hat{x}_{\alpha^0}$, leading to the point A of the figure and since $h_{0.5}(A)>0$, we have $\alpha_B=\alpha^0$, $\hat{x}_{\alpha_B}\equiv A$, $\alpha_A=f(A)=1.07$. Next we determine $\hat{x}_{\alpha_A}$ and the point I of the figure ($\hat{x}_{\alpha_A}\equiv I$) is obtained, with $h_{1.07}(I)\neq 0$;

-According to "STEP 0", once $f_2(\hat{x}_{\alpha_A})\neq f_2(\hat{x}_{\alpha_B})$ we proceed to "STEP 1".

-Since $(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B}))\neq\{\alpha_A,\alpha_B\}$, by applying "STEP 1", we put $\alpha^1=\min\{9/15, f(A), f(I)\} = 9/15$ and $\hat{x}_{\alpha^1}$ is determined (see figure 3).

-As the figure 3 ilustrates, it may be concluded that $\hat{x}_{\alpha^1}\equiv J$ ($J'\equiv(f_2(\hat{x}_{\alpha^1}), f_1(\hat{x}_{\alpha^1}))$) and since $h_{\alpha^1}(\hat{x}_{\alpha^1})<0$ and $h_{\alpha^1}(\hat{x}_{\alpha^1})\neq h_{\alpha^1}(\hat{x}_{\alpha_A})$ α_A becomes α^1 ($\alpha_A\leftarrow\alpha^1$) and $\hat{x}_{\alpha_A}$ becomes $\hat{x}_{\alpha^1}$ (that is $\hat{x}_{\alpha_A}\leftarrow J$), according to the algorithm, and we proceed to STEP 2.

fig. 3



-Since $f_2(J)=f_2(I)$ [14] we determine δ, according to fomula (1) obtaining a value greater than the admissible maximum error, thus returning the process to "STEP 1". Then we put:

$\alpha^2=\min\{(f_1(J)-f_1(I))/(f_2(J)-f_2(I)), f(J), f(I)\}=f(J)=37/41$

and since $(f_1(J)-f_1(I))/(f_2(J)-f_2(I))\notin\{\alpha_A,\alpha_B\}$ $\hat{x}_\alpha 2$ is determined.

-As the last figure illustrates, $\hat{x}_\alpha 2=J$ (that is $(f_2(\hat{x}_\alpha 2), f_1(\hat{x}_\alpha 2))=J'$) and this time we have $h_\alpha 2(\hat{x}_\alpha 2)=0$. Therefore we may conclude that $J=X^*$.

5. Computational Experiments

-Several computational experiments were carried out in order to compare the efficiency of the different methods referred in the begining of this paper. For this purpose we have generated several problems by using a pseudo-random number generator and we have determined, for each method, the average values for the number of pivoting simplex operations, the number of iterations and the processing times needed for obtaining the optimal solutions.

-With this test we solved 120 linear fractional programming problems with the following characteristics:

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[14] See table of the last page.

20 decision variables;
20 contraints defined by inequalities;
All the coefficients of the decision matrix have absolute values in]0, 10], 20% of which are negative;
All the independent terms were obtained from the expression:

$$b_i = \sum_{j=1}^{20} \frac{a_{ij}}{2} \quad \text{for } i=1,2, \ldots ,20$$

The coefficients of the numerator function of the objective function, were generated in [-1000, 1000];
The coefficients of denominator function of the objective function, were generated in]0, 2];
25% of the generated constraints are of the type $\leq$;
For the bicriteria approach method, for the Isbell et al's method and for the MODBIL, descrived in [14], we used for α^0 the value suggested by Bitran et al [5] [15] (in general, this value leads to good results) and the upper bound of the admissible error in the stoping--criteria has the value 0.005;
The initial parâmeters for the MODBIL, are obtained in the same way as in the bicriteria approach method (by application of the step 0 of the algorithm);
For the tests concerning the bicriteria approach method and MODBIL, where the determination of the inicial parameters α_A and α_B with $\eta(\alpha_A)>0$ and $\eta(\alpha_B)<0$ was made according to step 0 of the algorithm, we used k=2.

-These experiments, were made in conditions similar to those required by Bitran [11] with respect to the characteristics of the generated problems, but involving a greater number of problems and without the aim of studying the correlations between the obtained values and the geometric properties of the problems, the exclusive goal being to determine the average values and respective standard deviation.

-The tests were made in a microcomputer PC8-MCA, with the algorithms implemented in pascal and using a turbopascal compiler 3.01A of Borland Inc.

-Having in mind to improve the efficiency of the proposed method we introduce in the algorithm the following modification in the determination of the parameter, α^i, in step 1 of the algorithm:

[15] $\alpha^0 = \langle \nabla f_1, \nabla f_2 \rangle / \langle \nabla f_2, \nabla f_2 \rangle$.

-Let $\alpha_{AB}=(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B}))$ and $f(\bar{x})$ such that $f_1(\bar{x})-\alpha_A f_2(\bar{x})=\hat{h}_{\alpha_A}$ and $f_1(\bar{x})-\alpha_B f_2(\bar{x})=\hat{h}_{\alpha_B}$. The choice of the parameter, α^i, is done according to the procedure:

If $\min\{\alpha_{AB}, f(\hat{x}_{\alpha_A}), f(\hat{x}_{\alpha_B})\}=\alpha_{AB}$ then

$$\alpha^i=\alpha_{AB}$$

else

$$\alpha^i=\lambda(i)f(\bar{x})+(1-\lambda(i))\min\{f(\hat{x}_{\alpha_A}), f(\hat{x}_{\alpha_B})\};$$

with $\lambda(i)=\frac{1}{i}$. Other choices suggested in [14], such as $\lambda(i)=1/(i+1)$ e $\lambda(i)=1/2^i$, led to the same results and we suppose that these are of interest only in problems of dimension greater than the one allowed by the available equipment in our study.

-This procedure was suggested by the acceleration techniques of the bisection method when applied to the calculations of the root of equation, $\eta(\alpha)=0$, as described in [14] namely in what concerns MODBIL and MODBIN.

-The obtained results are shown in the following table:

Average numbers (μ) and standard deviations (σ) of the numbers of pivotting operations, number of iterations and processing time in seconds						
Method	pivotting operations μ	pivotting operations σ	iterations μ	iterations σ	processing time μ	processing time σ
Charnes et al	66	20	--	--	57.9	16
Martos	40	6.8	--	--	33.8	5.5
Frankel et al	67	15	4.7	2.5	46.5	8
Isbell et al	47	9.5	3.3	0.6	36.2	6.3
MODBIL	50	9.2	2.9	0.4	39.3	5.6
Bicriteria Approach	49	10	3	0.5	38	6.6

6. Conclusions.

-In what concerns the methods which enable to evaluate the error incurred when the current solution is approximated by the iterated point, we may conclude, from the computational tests, that there is a certain competitivity between MODBIL and the proposed method.

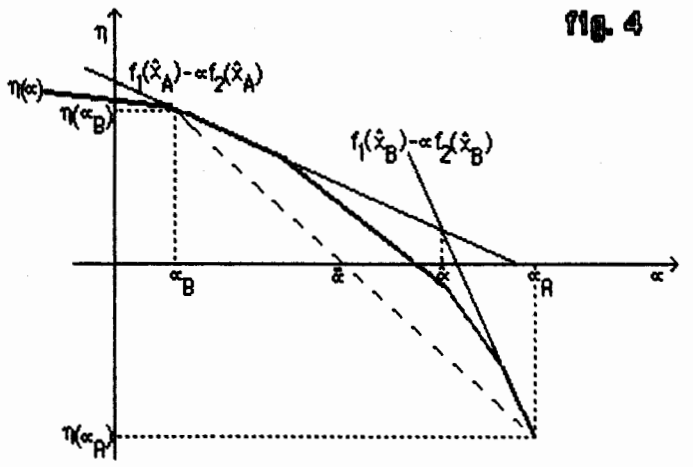
-The inclusion of MODBIL in the tests (chosen among the methods described in [14]) stems from the fact that it is the method closer the one presented here and from the fact that, according to the

tests presented in [14], it leads to good results. The use of polinomial interpolation - such as in ITBIN which is the method recommended in [14] as being the fastest one - could also be included in the proposed algorithm. However we think that this only would have significant consequences in problems of dimension which we were not able to test with the available computational means.

-MODBIL, as well as the presented method, leads to the same upper bound for the error of the current solution. This may be easily concluded by comparing figure 4, where the function $\eta(\alpha)$ is represented, with figure 1. Indeed, from this comparison we can establish the relations:

$$\eta(\alpha_A)=\hat{h}_{\alpha_A};\quad \eta(\alpha_B)=\hat{h}_{\alpha_B};\quad \bar{\alpha}=f(\bar{x});\quad \hat{\alpha}=\frac{f_1(\hat{x})_{\alpha_A}-f_1(\hat{x})_{\alpha_B}}{f_2(\hat{x})_{\alpha_A}-f_2(\hat{x})_{\alpha_B}}$$

Fig. 4



- Although our method has been presented for the linear case, it can be easily extended to the nonlinear fractional programs dealt with in [13] and [14].

-As the computational experiments shows, the method introduced by Martos requires, in most cases, fewer pivoting simplex operations than any of the other methods here referred, being obviously more rapid. However it should be referred that this method, although being faster, only should be used when the error associated with the determined solution is not relevant to the application under consideration.

-Although the Charnes and Cooper's algorithm is algorithmic equivalent to the one of Martos, according to the proof by Wagner and Yuan [10], the first one has in most cases values of pivoting simplex operations and processing times greater than the second, as a result of the changes in the decision space. These differences are essentially originated by the suplementary effort that the determination of the first admissible solution leads to.

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Appendix

Lemma 4.1

Let (P_{α_A}) and (P_{α_B}) such that $\hat{h}_{\alpha_A}<0$, $\hat{h}_{\alpha_B}>0$ and $f_2(\hat{x}_{\alpha_A})\neq f_2(\hat{x}_{\alpha_B})$, then:

$$f_1(\hat{x}_{\alpha_B})\leqq f_1(\hat{x}_{\alpha_A}) \wedge f_2(\hat{x}_{\alpha_B})<f_2(\hat{x}_{\alpha_A})$$

proof:

Let $x^*\in X^*$, by theorem 3, we'll have:

$$\hat{h}_{\alpha_A}<0 \Rightarrow f_1(x^*)\leqq f_1(\hat{x}_{\alpha_A}) \wedge f_2(x^*)\leqq f_2(\hat{x}_{\alpha_A})$$

$$\hat{h}_{\alpha_B}>0 \Rightarrow f_1(\hat{x}_{\alpha_B})\leqq f_1(x^*) \wedge f_2(\hat{x}_{\alpha_B})\leqq f_2(x^*),$$

so $f_1(\hat{x}_{\alpha_B})\leqq f_1(\hat{x}_{\alpha_A}) \wedge f_2(\hat{x}_{\alpha_B})\leqq f_2(\hat{x}_{\alpha_A})$, since by hypothesis $f_2(\hat{x}_{\alpha_A})\neq f_2(\hat{x}_{\alpha_B})$ we have $f_2(\hat{x}_{\alpha_B})<f_2(\hat{x}_{\alpha_A})$.

□

Lemma 4.2

Let (P_{α_A}) and (P_{α_B}) such that $\hat{h}_{\alpha_A}<0$, $\hat{h}_{\alpha_B}>0$, $f_2(\hat{x}_{\alpha_A})\neq f_2(\hat{x}_{\alpha_B})$ and $\bar{\alpha}=\min\{(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})), f(\hat{x}_{\alpha_A}), f(\hat{x}_{\alpha_B})\}$.

Also assume $(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B}))\neq\{\alpha_A,\alpha_B\}$.

Then $$\alpha_B < \bar{\alpha} < \alpha_A.$$

proof:

$\hat{h}_{\alpha_c} \leq f_1(x)-\alpha_c f_2(x) \quad \forall x \in X \quad \forall c \in \{A,B\}$, so in particular we have:

$$f_1(\hat{x}_{\alpha_B})-f_1(\hat{x}_{\alpha_A}) \leq \alpha_B (f_2(\hat{x}_{\alpha_B})-f_2(\hat{x}_{\alpha_A}))$$

and $$f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}) \leq \alpha_A (f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})).$$

Since by Lemma 4.1 $f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})>0$, we have:

$\alpha_B \leq (f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})) \leq \alpha_A$ and since by hypothesis $(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})) \notin \{\alpha_B,\alpha_A\}$, we may conclude:

$$\alpha_B < (f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})) < \alpha_A \quad (*)$$

From $(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})) < \alpha_A$ we have $\bar{\alpha}<\alpha_A$, so we'll only need to prove that $\alpha_B<\bar{\alpha}$.

$\hat{h}_{\alpha_B}>0 \Leftrightarrow f(\hat{x}_{\alpha_B})>\alpha_B \quad (**)$, have into account $(*)$, if $f(\hat{x}_{\alpha_A})>\alpha_B$ then $\alpha_B<\bar{\alpha}$.

Let us suppose that $f(\hat{x}_{\alpha_A}) \leq \alpha_B$, then:

$$f_1(x)-\alpha_B f_2(x) \leq f_1(x)-f(\hat{x}_{\alpha_A}) f_2(x) \quad \forall x \in X,$$

and in particular we have $h_{\alpha_B}(\hat{x}_{\alpha_A}) \leq f_1(\hat{x}_{\alpha_A})-f(\hat{x}_{\alpha_A}) f_2(\hat{x}_{\alpha_A})=0$ and in view of $\hat{h}_{\alpha_B} \leq h_{\alpha_B}(\hat{x}_{\alpha_A})$, we may conclude against the hypothesis that $\hat{h}_{\alpha_B} \leq 0$. So $\alpha_B < f(\hat{x}_{\alpha_A})$ and from this conclusion, from $(*)$ and $(**)$, it finally results that $\alpha_B<\bar{\alpha}$.

□

Lemma 4.3

Let (P_{α_A}) and (P_{α_B}) such that $\hat{h}_{\alpha_A}<0$, $\hat{h}_{\alpha_B}>0$, $f_2(\hat{x}_{\alpha_A}) \neq f_2(\hat{x}_{\alpha_B})$ and $\bar{\alpha}=\min((f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})), f(\hat{x}_{\alpha_A}), f(\hat{x}_{\alpha_B}))$.

Suppose also that $(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})) \notin \{\alpha_A,\alpha_B\}$.

Then:

$\forall x \in X$: $h_{\bar{\alpha}}(x) \leq h_{\bar{\alpha}}(\hat{x}_{\alpha_c}) \quad \forall c \in \{A,B\}$

$$\Rightarrow f_1(\hat{x}_{\alpha_B}) \leq f_1(x) \wedge f_2(x) \leq f_2(\hat{x}_{\alpha_A})$$

proof:

From the hypothesis we have:

$$f_1(\hat{x}_{\alpha_c})-f_1(x) \leq \alpha_c (f_2(\hat{x}_{\alpha_c})-f_2(x)) \quad \forall x \in X \quad \forall c \in \{A,B\} \quad (*)$$

Let $x \in X$: $f_1(x)-f_1(\hat{x}_{\alpha_c}) \leq \bar{\alpha}(f_2(x)-f_2(\hat{x}_{\alpha_c})) \quad \forall c \in \{A,B\}$ so we can concluded that:

$$0 \leq (\alpha_c-\bar{\alpha})(f_2(\hat{x}_{\alpha_c})-f_2(x)) \quad \forall c \in \{A,B\},$$

and for c=A, we get $0 \leq (\alpha_A - \bar{\alpha})(f_2(\hat{x}_{\alpha_A}) - f_2(x))$, and for c=B, we obtain $0 \leq (\alpha_B - \bar{\alpha})(f_2(\hat{x}_{\alpha_B}) - f_2(x))$. From Lemma 4.2 we know that $\alpha_B < \bar{\alpha} < \alpha_A$, therefore we have $\alpha_A - \bar{\alpha} > 0$ and $\alpha_B - \bar{\alpha} < 0$, so we may conclude that $f_2(\hat{x}_{\alpha_A}) - f_2(x) \geq 0$ and $f_2(\hat{x}_{\alpha_B}) - f_2(x) \leq 0$. From this conclusion and from (*), finally follows that $f_1(x_B) - f_1(x) \leq 0$.

□

Lemma 5.1

Let (P_{α_A}) and (P_{α_B}) such that $\hat{h}_{\alpha_A} < 0$, $\hat{h}_{\alpha_B} > 0$ and $f_2(\hat{x}_{\alpha_A}) \neq f_2(\hat{x}_{\alpha_B})$ then:

$$(f_1(\hat{x}_{\alpha_A}) - f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A}) - f_2(\hat{x}_{\alpha_B})) \in \{\alpha_A, \alpha_B\} \Rightarrow \{\hat{x}_{\alpha_A}, \hat{x}_{\alpha_B}\} \cap X^* \neq \emptyset.$$

proof:

Let $x^* \in X^*$, Theorem 3, establishes:

$$\hat{h}_{\alpha_A} < 0 \Rightarrow f_1(x^*) \leq f_1(x_A) \wedge f_2(x^*) \leq f_2(x_A)$$

$$\hat{h}_{\alpha_B} > 0 \Rightarrow f_1(x_B) \leq f_1(x^*) \wedge f_2(x_B) \leq f_2(x^*)$$

Let $\alpha_c = (f_1(\hat{x}_{\alpha_A}) - f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A}) - f_2(\hat{x}_{\alpha_B}))$, with $c \in \{A,B\}$, i.e, $h_{\alpha_c}(\hat{x}_{\alpha_A}) = h_{\alpha_c}(\hat{x}_{\alpha_B})$, with $c \in \{A,B\}$.

(i) If c=A then $\hat{h}_{\alpha_A} = h_{\alpha_c}(x_B) < 0$ and by Theorem 4 we have $f_1(x^*) \leq f_1(\hat{x}_{\alpha_B})$ and $f_2(x^*) \leq f_2(\hat{x}_{\alpha_B})$ and so we have:

$$f_1(\hat{x}_{\alpha_B}) \leq f_1(x^*) \leq f_1(\hat{x}_{\alpha_B}) \wedge f_2(\hat{x}_{\alpha_B}) \leq f_2(x^*) \leq f_2(\hat{x}_{\alpha_B})$$

or $f_1(\hat{x}_{\alpha_B}) = f_1(x^*) \wedge f_2(\hat{x}_{\alpha_B}) = f_2(x^*)$, therefore $\hat{x}_{\alpha_B} \in X^*$.

(ii) For c=B, we'll conclude in an identical way that $\hat{x}_{\alpha_A} \in X^*$.

Nevertheless in any case we'll have $\{\hat{x}_{\alpha_A}, \hat{x}_{\alpha_B}\} \cap X^* \neq \emptyset$.

□

Lemma 5.2

Let $x^* \in X^*$, $x_A, x_B \in X$: $f_1(x_B) \leq f_1(x^*) \leq f_1(x_A) \wedge f_2(x_B) \leq f_2(x^*) \leq f_2(x_A)$ then:

$$f_2(x_A) = f_2(x_B) \Rightarrow x_B \in X^*$$

proof:

From $1/f_2(x_A) \leq 1/f_2(x^*) \leq 1/f_2(x_B)$ and $f_2(x_A) = f_2(x_B)$ we obtain

$$1/f_2(x_B) = 1/f_2(x^*) \Rightarrow f_1(x_B)/f_2(x_B) \leq f_1(x^*)/f_2(x^*) \Leftrightarrow x_B \in X^*$$

□

Lemma 5.3

Let (P_{α_A}) (P_{α_B}), such that $\hat{h}_{\alpha_A}<0$, $\hat{h}_{\alpha_B}>0$ and $f_2(\hat{x}_{\alpha_A})\neq f_2(\hat{x}_{\alpha_B})$.

Let $\bar{\alpha}=\min\{(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})), f(\hat{x}_{\alpha_A}), f(\hat{x}_{\alpha_B})\}$ and $\hat{h}_{\bar{\alpha}}\neq 0$.

Then $$\hat{h}_{\bar{\alpha}}=h_{\bar{\alpha}}(\hat{x}_{\alpha_A}) \Leftrightarrow h_{\bar{\alpha}}=h_{\bar{\alpha}}(\hat{x}_{\alpha_B})$$

proof:

($\Rightarrow$)

Suppose that $\hat{h}_{\bar{\alpha}}=h_{\bar{\alpha}}(\hat{x}_{\alpha_A})$, then we have:

$$f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B})\leq\bar{\alpha}(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})).$$

Since the hypothesis of Lemma 4.1 are held, we can conclude that $f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})>0$ and so we have:

$$(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B}))\leq\alpha^i$$

and hence:

$$\bar{\alpha}=(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})) \Leftrightarrow h_{\bar{\alpha}}(\hat{x}_{\alpha_A})=h_{\bar{\alpha}}(\hat{x}_{\alpha_B})$$

($\Leftarrow$)

If $\bar{\alpha}=\{f(\hat{x}_{\alpha_A})), f(\hat{x}_{\alpha_B})\}$ then $\hat{h}_{\bar{\alpha}}\leq 0$. Suppose that $\hat{h}_{\bar{\alpha}}=h_{\bar{\alpha}}(\hat{x}_{\alpha_B})$, then $\hat{h}_{\bar{\alpha}}\geq 0$ (since $\bar{\alpha}\leq f(\hat{x}_{\alpha_B}) \Leftrightarrow h_{\bar{\alpha}}(\hat{x}_{\alpha_B})\geq 0$) and since by hypothesis $\hat{h}_{\bar{\alpha}}\neq 0$, we may conclude that $\bar{\alpha}=\{f(\hat{x}_{\alpha_A})), f(\hat{x}_{\alpha_B})\}$ and so

$$\bar{\alpha}=(f_1(\hat{x}_{\alpha_A})-f_1(\hat{x}_{\alpha_B}))/(f_2(\hat{x}_{\alpha_A})-f_2(\hat{x}_{\alpha_B})).$$

Therefore $h_{\bar{\alpha}}(\hat{x}_{\alpha_A})=h_{\bar{\alpha}}(\hat{x}_{\alpha_B})$, thus completing the proof.

□

Lemma 5.4

Let (P_{α_A}) such that $\hat{h}_{\alpha_A}<0$ and let $\bar{\alpha}\in\mathbb{R}^+$ such that $\bar{\alpha}\leq f(\hat{x}_{\alpha_A})$ and $\hat{h}_{\bar{\alpha}}\neq 0$.

Then $$\hat{h}_{\bar{\alpha}}=h_{\bar{\alpha}}(\hat{x}_{\alpha_A}) \Rightarrow \hat{x}_{\alpha_A}\in X^*$$

proof:

If $\hat{h}_{\bar{\alpha}}=h_{\bar{\alpha}}(\hat{x}_{\alpha_A})$ then, since by hipothesis $\bar{\alpha}\leq f(\hat{x}_{\alpha_A}) \Leftrightarrow h_{\bar{\alpha}}(\hat{x}_{\alpha_A})\geq 0$ and $\hat{h}_{\bar{\alpha}}\neq 0$, we have $\hat{h}_{\bar{\alpha}}>0$ and by Theorem 3 we conclude that $f_1(\hat{x}_{\alpha_A})\leq f_1(x^*)$ and $f_2(\hat{x}_{\alpha_A})\leq f_2(x^*)$. On the other hand we know, by hypothesis that $\hat{h}_{\alpha_A}<0$. So from the same theorem we may conclude that $f_1(x^*)\leq f_1(\hat{x}_{\alpha_A})$ and $f_2(x^*)\leq f_2(\hat{x}_{\alpha_A})$. Therefore $x_A\in X^*$.

□

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SIMPLE HOMOGENEOUS PREDICTORS FOR FRECHET DISTRIBUTIONS

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ABSTRACT

Assume that a large sample of size n is available from a population having a Fréchet distribution for maxima. From that sample we derive simple homogeneous predictors for the largest value of the next m observations obtained from the same population. The predictors are obtained by splitting the ordered sample in two blocks and using their averages. High asymptotic efficiency of each simple predictor with respect to the best homogeneous predictor is obtained and is independent of the number m of the next observations. Homogeneous prediction region and the best homogeneous predictor for a censored right sample are also derived.

1. Introduction

Let $(X_1, X_2, ..., X_n)$ be an observed sample obtained from a population with known distribution, except possibly for some parameters. The prediction problem is, from the observed sample, to obtain a statistic forecasting, a function of the next m observations $(X_{n+1}, X_{n+2}, ..., X_{n+m})$.

The measure traditionally used in the statistical theory of prediction is the mean square error between the predictor $f(X_1, X_2, ..., X_n)$ and the value to be predicted $Z = \psi(X_{n+1}, ..., X_{n+m})$, $M.S.E.=E[(f - Z)^2]$. This is the criterion for choice of the predictor that we will use.

Tiago de Oliveira (1966,1968) gave the general solution for the best (least-squares) quasi-linearly invariant predictor and lower bounds for the mean square error of prediction. Tiago de Oliveira and Littauer (1976) formulated quasi-linearly invariant predictors for the minima of the Weibull and the maxima of the Gumbel distributions.

In the present work we are going to consider only homogeneous prediction for the Fréchet distribution .

Recall that a statistical function φ is said to be homogeneous if

$$\varphi(\beta X_1, ..., \beta X_n) = \beta\varphi(X_1, ..., X_n) \qquad (\beta > 0).$$

As some examples of homogeneous statistics we can consider the largest value, the minimum value, the average and more generally linear functions of the order statistics.

The homogeneous function we intend to predict is $Z = \max(X_{n+1}, ..., X_{n+m})$ owing to its interest in practical applications.

2. Homogeneous prediction

2.1. General results

From the observed sample $(X_1, X_2, ..., X_n)$, with known Fréchet distribution with a scale parameter, Neves (1987) obtained the general solution for the best homogeneous predictor as

$$f(x_1, ..., x_n) = \frac{\int_0^{+\infty} d\beta \beta^{n-1} \overline{\mathcal{L}}(\beta x_1, ..., \beta x_n) \mu_m(\beta x_1, ..., \beta x_n)}{\int_0^{\infty} d\beta \beta^n \overline{\mathcal{L}}(\beta x_1, ..., \beta x_n)} \tag{1}$$

where $\overline{\mathcal{L}}(x_1, ..., x_n)$ and $\mu_m(x_1, ..., x_n)$ denote respectively, the marginal likelihood of $(X_1, ..., X_n)$ and the conditional mean of Z.

Recall that the Fréchet distribution for largest values, with known shape and location parameters, $\kappa = \kappa_0 > 0$ and $\lambda = \lambda_0$ ($\lambda_0 = 0$ for simplicity), is

$$F(x; \delta) = \begin{cases} \exp(-(\frac{x}{\delta})^{-\kappa_0}) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{2}$$

Thus if $X_1, X_2, ..., X_n$ are n i.i.d. random variables following (2) the homogeneous predictor assumes the form

$$f_n = f(x_1, ..., x_n) = \frac{\mu_m \Gamma(n)}{(nM_n)^{1/\kappa_0} \Gamma(n - 1/\kappa_0)}$$

with

$$\mu_m = m^{1/\kappa_0} \Gamma(1 - 1/\kappa_0) \quad \text{and} \quad M_n = \frac{1}{n} \sum_1^n x_i^{-\kappa_0} \quad (\kappa_0 > 1).$$

In our previous paper, Neves (1987), we proved that

$$\sqrt{n}\kappa_0 (\frac{f_n}{\mu_m} - 1) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

2.2. Simple homogeneous predictors

In Neves (1987) we have considered the simple homogeneous predictors:

$$a(\kappa_0, m)\overline{X}, \quad b(\kappa_0, m)Q \quad \text{and} \quad c(\kappa_0, m)M_n^{'}$$

where $a(\kappa_0, m)$, $b(\kappa_0, m)$ and $c(\kappa_0, m)$ are coefficients that minimize the mean square error of each of the predictors about the conditional mean μ_m, $D_{n,m}^2$ [1] and $\overline{X}$, Q and $M_n^{'}$ are the sample average, the sample quantile for probability p and the sample largest value, respectively.

We have obtained a zero efficiency for $c.M_n^{'}$ (let us note that $\text{var}(M_n^{'}) = n^{2/\kappa_0}.(\Gamma(1-2/\kappa_0) - \Gamma^2(1-1/\kappa_0))$). About the two other predictors we have obtained reasonable efficiencies, independent of m.

A short table of some of those values for the predictor $a.\overline{X}$

κ_0	3	4	5	10
$eff(a_1.\bar{x})$	0.24	0.34	0.41	0.51

The asymptotic efficiency of $b.Q$ is maximum for the quantile of probability $p = 0.2$; its value is 0.6476. For this predictor the efficiency is independent of κ_0 and m.

The papers of Kubat (1975,1982,1984), Kubat and Epstein (1980) and Hüsler and Schüpbach (1986) lead us to consider the simple homogeneous predictors

$$L_1 = a_1X_1 + a_2X_2 \quad \text{and} \quad L_2 = b_1Q_1 + b_2Q_2$$

[1] Let us note that

$$M.S.E.(f) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} (Z-f)^2 \mathcal{L}(x_1, \dots, x_n; z)dx_1 \dots dx_n dz =$$

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} (Z-\mu_m)^2 \mathcal{L}(x_1, \dots, x_n; z)dx_1 \dots dx_n.dz + \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} (f-\mu_m)^2 \overline{\mathcal{L}}(x_1, \dots, x_n; z)dx_1 \dots dx_n$$

$$= \sigma_{n.m}^2 + D_{n.m}^2(f)$$

and, as $\sigma_{n,m}^2$ is a constant, minimizing $M.S.E.(f)$ is equivalent to minimize $D_{n,m}^2(f)$

where $\overline{X}_1$ and $\overline{X}_2$ are the averages of two contiguous blocks of a convenient splitting the sample of the ordered observations, $a_1 = a_1(\kappa_0, m)$; $a_2 = a_2(\kappa_0, m)$; $b_1 = b_1(\kappa_0, m)$ and $b_2 = b_2(\kappa_0, m)$ are determined like for the previous simple predictors.

The criterion for selecting the two blocks is that one of maximizing the asymptotic efficiency of L_1 and/or L_2 about the best homogeneous predictor f_n.

Let us denote by $X_{(1)} \leq ... \leq X_{(n)}$ the order statistics corresponding to $X_1, ..., X_n$ obtained from (2).

Let $X_{(1)} \leq ... \leq X_{(r)}$ and $X_{(r+1)} \leq ... \leq X_{(n)}$ be the two blocks, with $r = [np]$, $0 < p < 1$ and $\overline{X}_1 = \frac{\sum_1^r X_{(i)}}{r}$ and $\overline{X}_2 = \frac{\sum_{r+1}^n X_{(i)}}{n-r}$.

The asymptotic distribution theory for the averages of the blocks of order statistics can be found conveniently summarized in Kubat (1982), from the results of Chernoff *et al.* (1967).

$Q_1 = X_{([np]+1)}$ and $Q_2 = X_{([nq]+1)}$, $0 < p < q < 1$, are the sample quantiles, p and q being also selected by maximization of the asymptotic efficiency of L_2 about f_n.

From Kubat results (1982) we have, for large n,

$$\overline{X} \sim \mathcal{N}(\mu, \frac{1}{n}V)$$

with $\overline{X}^T = [\overline{X}_1 \ \ \overline{X}_2]$, $\mu^T = [\mu_1 \ \ \mu_2]$ and

$$V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

with

$$\mu_1 = \frac{1}{p}\int_0^{\chi_p} x f(x)dx; \quad \mu_2 = \frac{1}{1-p}\int_{\chi_p}^{\infty} x f(x)dx;$$

$$\sigma_1^2 = \frac{1}{p}[\frac{1}{p}\int_0^{\chi_p} x^2 f(x)dx - \mu_1^2 + (\chi_p - \mu_1)^2(1-p)];$$

$$\sigma_2^2 = \frac{1}{1-p}[\frac{1}{1-p}\int_{\chi_p}^{+\infty} x^2 f(x)dx - \mu_2^2 + (\mu_2 - \chi_p)^2 p];$$

$$\sigma_{12} = (\mu_2 - \chi_p)(\chi_p - \mu_1).$$

Thus, the mean square error $D^2_{n,m}$ of L_1 can be written as

$$D^2_{n,m}(L_1) = E[(a_1\overline{X}_1 + a_2\overline{X}_2 - \mu_m)^2] =$$

$$= var(a_1\overline{X}_1 + a_2\overline{X}_2) + (a_1\mu_1 + a_2\mu_2 - \mu_m)^2 = a^T V a + (a^T\mu - \mu_m)^2$$

with $a^T = [a_1 \;\; a_2]$.

The limiting values of a that minimize the mean square error are

$$a_1 = \mu_m \frac{\mu_1\sigma_2^2 - \mu_2\sigma_{12}}{\mu_2^2\sigma_1^2 + \mu_1^2\sigma_2^2 - 2\mu_1\mu_2\sigma_{12}} \quad \text{and} \quad a_2 = \mu_m \frac{\mu_2\sigma_1^2 - \mu_1\sigma_{12}}{\mu_2^2\sigma_1^2 + \mu_1^2\sigma_2^2 - 2\mu_1\mu_2\sigma_{12}}$$

being then

$$eff(L_1) = \frac{var(f_n)}{var(a_1\overline{X}_1 + a_2\overline{X}_2)} \sim \frac{\mu_m^2}{\kappa_0^2 . a^T V a}.$$

The asymptotic efficiency is also independent of m and much better than that one of the previous simple predictor based on the sample mean.

For $m = 4$ (for other values of m only the coeficients a are different) a short table of the efficiency of L_1 is

κ_0	p	a_1	a_2	eff
2.5	.30	3.1039	.0491	.9120
3.0	.37	2.4399	.0770	.9214
4.0	.36	1.8324	.1058	.9323
5.0	.36	1.5559	.1168	.9383
6.0	.35	1.3954	.1296	.9422
8.0	.35	1.2259	.1363	.9468
10.0	.34	1.1300	.1470	.9494

Let us note that for the different values of κ_0, the splitting in blocks is made about the quantile $p = .36$.

Now we will go to consider the predictor

$$L_2 = b_1 . Q_1 + b_2 . Q_2 \quad \text{with} \quad Q_1 = X_{([np]+1)} \quad \text{and} \quad Q_2 = X_{([nq]+1)};$$

Q_1 and Q_2 are jointly asymptotically distributed with bivariate normal distribution, Cramér (1946) and Mosteller (1946).

Denoting by $Q^T = [Q_1 \ Q_2]$ and $\chi^T = [\chi_p \ \chi_q]$ we have

$$Q \sim \mathcal{N}(\chi, \frac{1}{n}V) \quad \text{where} \quad V = \begin{bmatrix} A & C \\ C & B \end{bmatrix}$$

with

$$A = \frac{p(1-p)}{f^2(\chi_p)}; \quad B = \frac{q(1-q)}{f^2(\chi_q)}; \quad \text{and} \quad C = \frac{p(1-q)}{f(\chi_p)f(\chi_q)}.$$

The minimun mean square error is attained for the limiting values

$$b_1 = \mu_m \frac{\chi_p.B - \chi_q.C}{A.\chi_q^2 + B\chi_p^2 - 2C\chi_p\chi_q} \quad \text{and} \quad b_2 = \mu_m \frac{\chi_q.A - \chi_p.C}{A.\chi_q^2 + B\chi_p^2 - 2C\chi_p\chi_q}.$$

The asymptotic relative efficiency is independent of m and of κ_0 and its maximum value is .82016 for the quantiles $p = .07$ and $q = .36$.

3. Homogeneous prediction region for the Fréchet distribution

Let us recall that a prediction region is determined by an indicator function $\varphi(x_1, x_2, ..., x_n; z)$ taking the values 0 or 1 depending on $z \notin R(x_1, x_2, ..., x_n)$ or $z \in R(x_1, x_2, ..., x_n)$.

Tiago de Oliveira (1966) has derived the prediction region in the quasi-linear case; in the homogeneous case φ takes the value 1 on the set of $(x_1, ..., x_n; z)$ where

$$\int_0^\infty d\beta\beta^{n-1}\mathcal{L}(\beta x_1, ..., \beta x_n, \beta z) \geq C\int_0^\infty d\beta\beta^{n-1}\overline{\mathcal{L}}(\beta x_1, ..., \beta x_n) \qquad (3)$$

where C is such that the prediction level ω satisfies

$$\int_R \delta^{-(n+1)}\mathcal{L}(\frac{x_1}{\delta}, \frac{x_2}{\delta}, ..., \frac{x_n}{\delta}, \frac{z}{\delta})dx_1...dx_2dz = \omega.$$

For Fréchet distributions, the prediction region is then:

$$\frac{\kappa_0 m z^{-\kappa_0-1}\Gamma(n+1+1/\kappa_0)}{(nM + mz^{-\kappa_0})^{n+1+1/\kappa_0}} \geq C\frac{(n-1)!}{n^n M^n} \qquad (4)$$

where $M = \frac{1}{n}\sum x_i^{-\kappa_0}$.

Let us consider $\phi = \frac{mx^{-\kappa_0}}{nM}$. The region (4) can be written as

$$\frac{\phi^{1+1/\kappa_0}}{(1+\phi)^{n+1+1/\kappa_0}} \geq C'.$$

with $C' = C\frac{(n-1)!}{\kappa_0\Gamma(n+1+1/\kappa_0)}$.

4. Homogeneous prediction for right-censored large samples

Let us suppose than only the first r observations $X_{(1)}, ..., X_{(r)}$, $(r = [np]$, $0 < p < 1)$, of a sample of size n obtained from (2), have been observed.

For the censored sample we have the marginal likelihood

$$\mathcal{L}(x_1, ..., x_r) = \kappa_0^r (x_1 ... x_r)^{-\kappa_0 - 1} \exp(-(x_1^{-\kappa_0} + ... + x_{r-1}^{-\kappa_0} + r.x_r^{-\kappa_0})) \frac{n!}{(n-r)!(r-1)!}(1 - e^{-x_r^{-\kappa_0}})^{n-r}$$

so the general solution for the best homogeneous predictor in the censored case is

$$f(x_1, ..., x_r) = \frac{\mu_m \int_0^{+\infty} d\beta \beta^{-r\kappa_0 - 1} \exp(-\beta^{-\kappa_0}(\sum_1^r x_i^{-\kappa_0} + (r-1)x_r^{-\kappa_0}))(1 - e^{-(\beta x_r)^{-\kappa_0}})^{n-r}}{\int_0^{+\infty} d\beta \beta^{-r\kappa_0} \exp(-\beta^{-\kappa_0}(\sum_1^r x_i^{-\kappa_0} + (r-1)x_r^{-\kappa_0}))(1 - e^{-(\beta x_r)^{-\kappa_0}})^{n-r}}. \tag{5}$$

Developing $(1 - e^{-(\beta x_r)^{-\kappa_0}})^{n-r}$ and by a change of variable in the integrals, we get the expression for the best homogeneous predictor for the Fréchet distribution based on a right censored sample

$$f(x_1, ..., x_r) = \frac{\mu_m \Gamma(r) \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} (\sum_{i=1}^{r-1} x_i^{-\kappa_0} + (r+j)x_r^{-\kappa_0})^{-r}}{\Gamma(r - 1/\kappa_0) \sum_{i=0}^{n-r} (-1)^j \binom{n-r}{j} (\sum_{i=1}^{r-1} x_i^{-\kappa_0} + (r+j)x_r^{-\kappa_0})^{-r+1/\kappa_0}}. \tag{6}$$

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SOME SILHOUETTE-BASED GRAPHICS FOR CLUSTERING INTERPRETATION

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ABSTRACT

Silhouettes were developed as a graphical display for nonhierarchical cluster analysis. They are based on the ratio between the tightness of a cluster and its separation from other clusters. A possible extension is to represent for each object both these characteristics in a two dimensional graph.
The same technique can also be used with fuzzy clustering, making use directly of the fuzzy membership functions to measure the tightness of the links of each object with its principal cluster and its neighbour.

1° Introduction

Visual representation has always been an important means of communication. Nowadays many other mathematical tools, such as analytical formulas and computers, are at the disposal of the researcher to describe phenomena in a precise way. However, graphical representation still possesses a very suggestive power that no other mathematical description is able to provide. The reason is that a graph yields a global view of the phenomena together with all the relations between its parts. This is clearly an advantage over most formal mathematical models.

No wonder that for cluster analysis, which is sometimes defined as the art of discovering groups in data, graphical representation is a much cherished tool. It may even be the main tool in examples where all objects can be represented in a two-dimensional space. In multidimensional situations clustering algorithms are necessary, but graphs are still very helpful to illustrate the results and to reveal some features which may be the start for a further investigation.

In hierarchical clustering, dendrograms [see e.g. ref. 1 to 5] represent the relations between the partitions at different levels, the merging sequence, and the level of each partition.

For nonhierarchical clustering, a representation by means of silhouettes was recently proposed by Rousseeuw [6]. Silhouettes are based on the ratio between the distances of an object to its own cluster and to its neighbour cluster.

In the present note, silhouettes will be extended in two directions: a two-dimensional representation for each object (Section 3) and a modification for fuzzy clusters, either as a one-dimensional (Section 4) or as a two-dimensional graph (Section 5). Some further considerations and conclusions are given in Section 6.

2° Recalling silhouettes

Silhouettes were developed by Rousseeuw [6] to evaluate the quality of a clustering allocation, independently of the clustering technique that was used. Only two streams of information are needed: the partition of the objects into a number of clusters (at least two) and the matrix of proximities between all objects.

The silhouettes are then defined as follows (we restrict ourselves to dissimilarities, although one could also use a collection of similarities between objects):

-let D(i,j) be the dissimilarity between objects i and j;

-let a(i) be the average dissimilarity of object i, which has been allocated to cluster A, to all other objects of the same cluster:

$$a(i) = \frac{\Sigma_j D(i,j)}{n_A - 1}$$

with $j \in A$ and n_A = number of objects in A. It is assumed that $n_A > 1$.

-let d(i,C) be the average dissimilarity of object i of cluster A to all objects of any cluster C, different from A; hence

$$d(i,C) = \frac{\Sigma D(i,j)}{n_C}$$

with $j \in C$ and n_C = number of objects in C.

-let b(i) be the minimum over all clusters C of d(i,C), corresponding to the neighbour cluster B (see Figure 1).

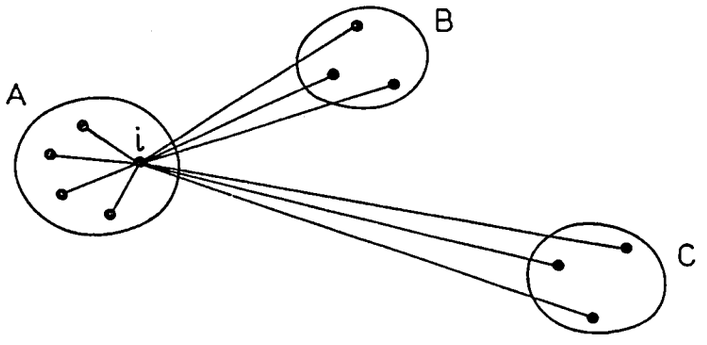

Fig.1: An illustration of the elements involved in the computation of s(i), where the object i belongs to cluster A (from [6]).

-let, for $n_A > 1$,

$$s(i) = 1 - \frac{a(i)}{b(i)} \qquad \text{if } a(i) < b(i)$$

$$= \frac{b(i)}{a(i)} - 1 \qquad \text{if } a(i) > b(i) \qquad (1)$$

for $n_A = 1$, $s(i) = 0$ by convention.

It can be seen that always

$$-1 \leq s(i) \leq 1. \qquad (2)$$

An s(i) near +1 means that the object i has a small average dissimilarity to objects of the same cluster and a high average dissimilarity to the neighbour cluster, and hence to all other clusters. A value near -1 expresses the opposite.

Having computed s(i) for each object of the data set, it is now possible to draw the silhouette of each cluster. For each object of that cluster, one draws a horizontal line with length proportional to s(i), pointing to the right whenever s(i) is positive and to the left otherwise (although this last part of the representation can be deleted as it is of less interest). All these lines are drawn below each other in decreasing order of magnitude. Each cluster has its own silhouette, the height of which is proportional to its number of objects whereas the width expresses its relative tightness.

Fig. 2 and 3 illustrate this technique on a set of objects consisting of two "natural" clusters. In Fig. 2 the natural clusters have effectively been found by some clustering technique. Because the clusters are fairly symmetric, so are both silhouettes. The largest values of s(i) correspond to objects at

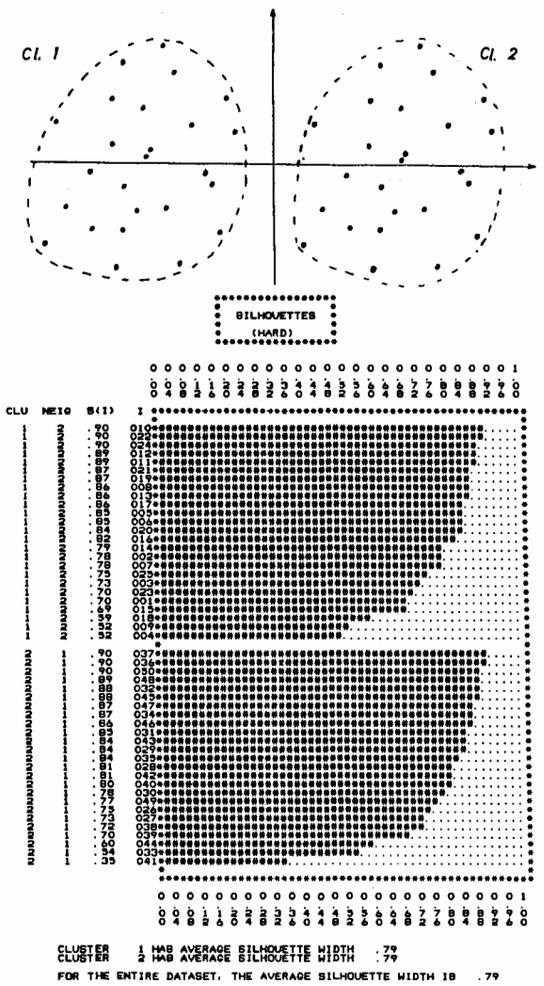

Fig.2: Silhouettes of basic model: 2 clusters.

the extremities of the set; the smallest values characterize objects near the interface between the clusters. The largest value is 0.90 for both clusters, and the smallest is 0.52 for the first cluster and 0.35 for the second. One can also calculate an average silhouette width for each cluster and for the entire data set; in our example all these values happen to be 0.79.

If a partition into three groups is performed (Fig.3), the first cluster remains unchanged whereas the second is split up in two parts. The silhouette of the first cluster is very similar to the one in Figure 2: not only the general shape is similar, but also the ordering of the objects. The s(i) values become slightly smaller because b(i), the average dissimilarity to the objects of the nearest of the other two clusters, is usually less than the average dissimilarity to the big cluster in Figure 2. This yields an average silhouette width of 0.75, as compared with 0.79 in Figure 2.

As for the two "half" clusters, the changes are of course more striking. Although for each object i the value a(i) is decreased, at the same time b(i) becomes smaller still, so s(i)=1-a(i)/b(i) decreases. This results in an average silhouette width of 0.50 for cluster 2 and 0.63 for cluster 3, as compared with 0.79 in Figure 2. The overall average silhouette width of all three clusters is 0.65, or about 20% less than in the case of two clusters. Therefore, the overall average silhouette width gives some indication about the "best" number of clusters.

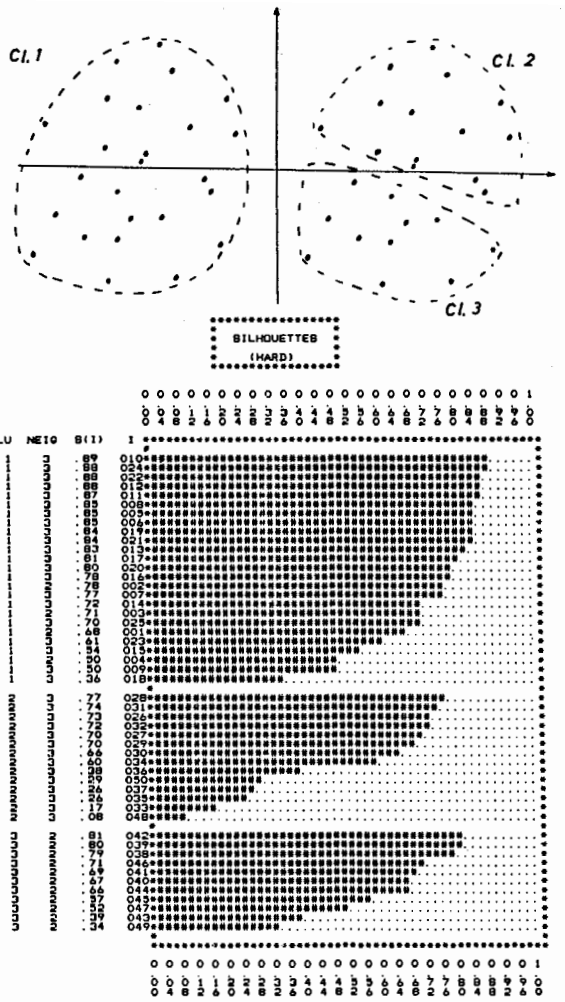

Fig.3: Silhouettes of basic model: 3 clusters.

3° Unfolding silhouettes in two dimensions

Silhouettes are based on the evaluation of two functions

for each object:

the "tightness" a(i)

the "separation" b(i).

Instead of calculating the ratio of these two functions, it is also possible to simply plot these functions in a two-dimensional graph, using, say, a(i) for the x-axis and b(i) for the y-axis.

As both a(i) and b(i) are always positive, only the first quadrant of the (x,y)-space is used. Looking for the relation between the s(i) values and the (a,b)-plot, it can be observed

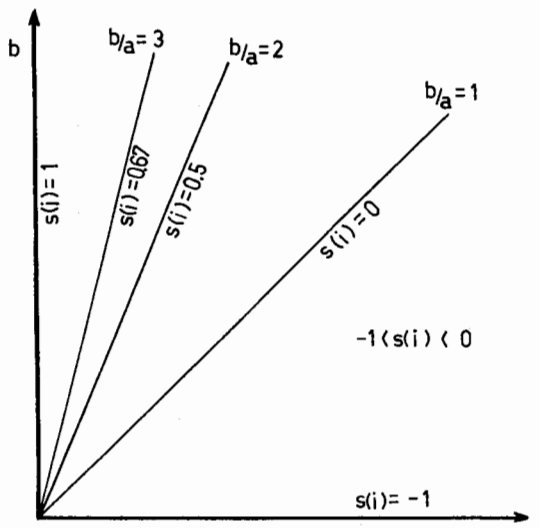

Fig.4: Relation between separation/tightness and silhouettes.

that all objects with the same s(i) values lie on a straight line, starting from the origin and satisfying one of the following equations:

$$b(i) = (1+s(i))\ a(i) \quad \text{if } -1 \leq s(i) \leq 0 \qquad (3)$$

$$b(i) = \frac{1}{1-s(i)}\ a(i) \quad \text{if } 0 \leq s(i) \leq 1 \qquad (4)$$

From these equations it can be seen that objects with s(i)=-1 will be represented by points on the a-axis. Objects with s(i)=0 correspond to the equation b(i)=a(i), and will be represented by points on the 45° line. Objects with negative s(i) will lie below that line, whereas objects with positive s(i) lie above it. Objects with s(i)=1 end up on the b-axis. These relations are represented in Fig. 4. It should be observed that a plot can be drawn for all the objects of a data set as well as for the objects of each cluster separately.

Fig. 5 and 6 show these plots for the example with two "natural" clusters discussed in the previous section. Fig. 5 is very typical of a good clustering allocation. The plots show a rather narrow concentration of the tightness a(i) and a much larger dispersion of the separation, with most objects having a b(i)/a(i) ratio larger than two. The only object with b(i)/a(i) smaller than two is located near both clusters. It almost forms a bridge between them, as can be deduced from the fact that a(i) has one of the largest and b(i) one of the smallest values.

In the three clusters case (Fig.6) things are clearly different. The first cluster still resembles that of the former

case, but the two remaining clusters have much smaller values of b(i), which in turn are much nearer to the a(i) values. This could be a first indication that these clusters should not have been separated.

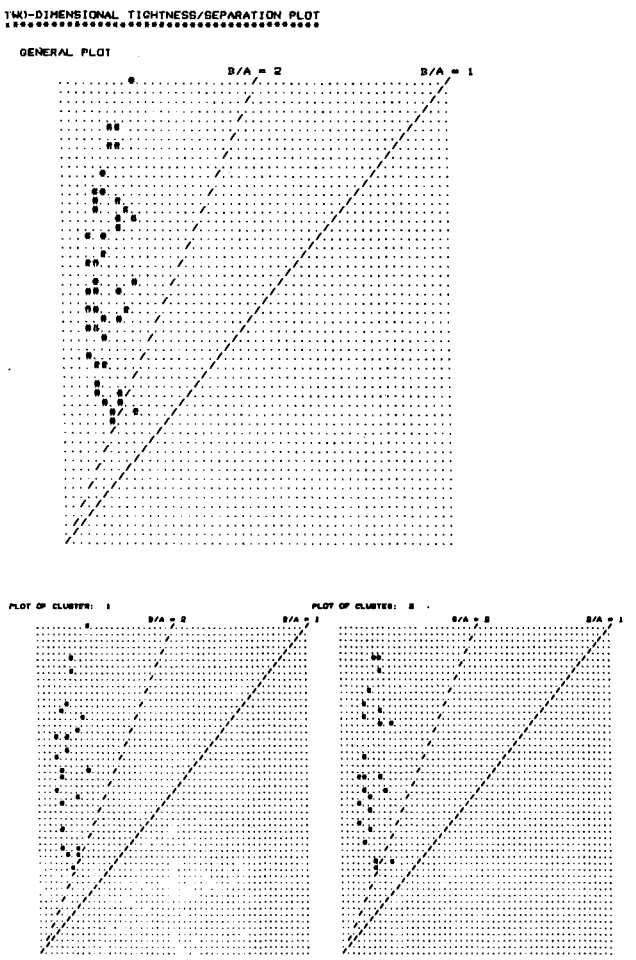

Fig.5: Basic model: two-dimensional hard representation of 2 clusters.

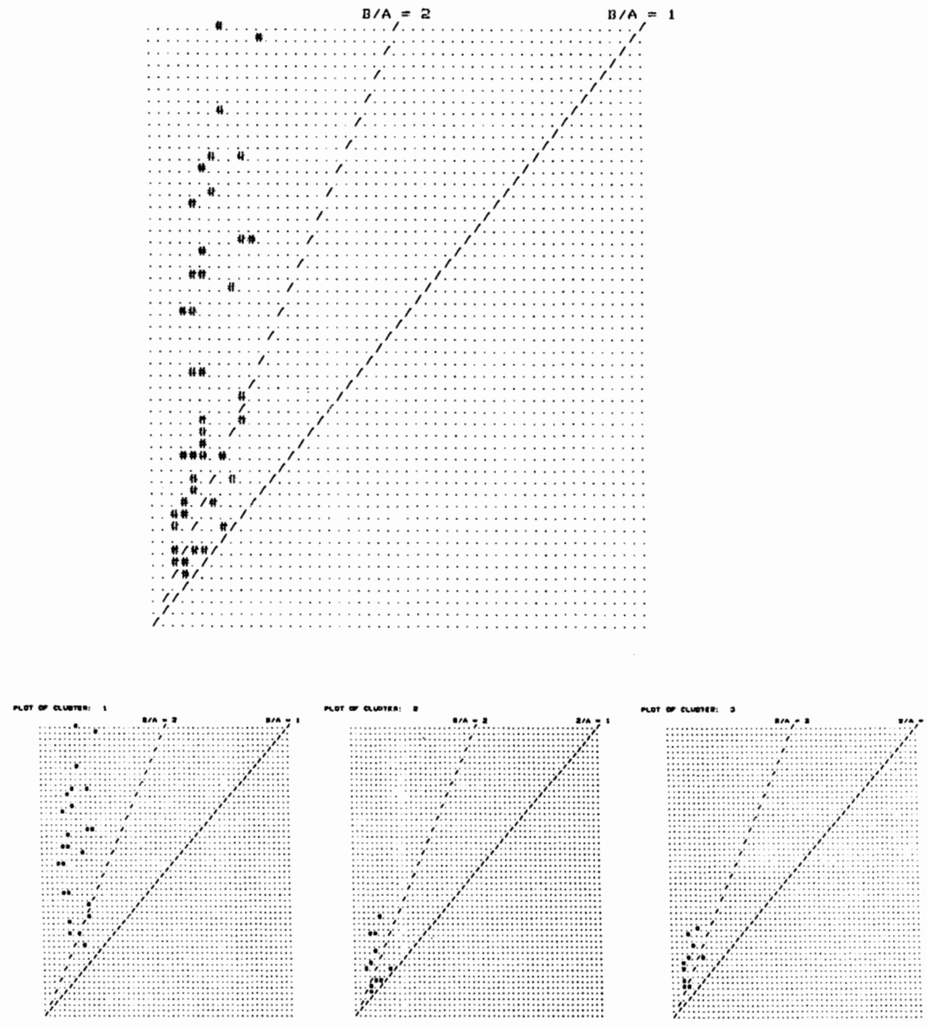

Fig.6: Basic model: two-dimensional hard representation of 3 clusters.

4° Using fuzzy membership functions

The goal of fuzzy clustering is to express, for each object, its relative membership to each cluster. Most fuzzy clustering algorithms [see e.g. ref.7] make use of average dissimilarities. By definition, the sum of membership values of each object to all clusters always equals one. It is also customary to consider the nearest hard classification, allocating each object to the cluster for which its fuzzy membership is largest. Therefore it is possible to define new "tightness" and "separation" factors based on membership functions, keeping in mind that the latter reflect similarity rather than dissimilarity:

$$a(i) = 1 - u^{\circ}(i) \quad \text{with } u^{\circ}(i) = u(t_{\circ},i) = \max_{t} u(t,i) \qquad (5)$$

$$b(i) = 1 - u^{\circ\circ}(i) \quad \text{with } u^{\circ\circ}(i) = \max_{t \neq t_{\circ}} u(t,i) \qquad (6)$$

in which the membership functions must satisfy the relations:

$$u(t,i) \geq 0 \qquad \text{for all } i \text{ and } t$$

$$\sum_{t} u(t,i) = 1 \qquad \text{for all } i. \qquad (7)$$

From (5) and (6) we see that

$$u^{\circ}(i) \geq u^{\circ\circ}(i) \qquad (8)$$

and hence we always have

$$a(i) \leq b(i) \qquad (9)$$

resulting in $0 \leq s(i) \leq 1$, excluding the possibility of negative $s(i)$. Apart from this last aspect, the $s(i)$ behave similarly to what was seen in section 2. This is confirmed by Fig.7 which shows the fuzzy silhouette plot of the two-cluster example of that section: the general shape is very similar to that of Fig.2.

The only difference is that the fuzzy s(i) are generally a bit larger than the hard s(i) (which, of course, depends very much on the actual fuzzy algorithm used).

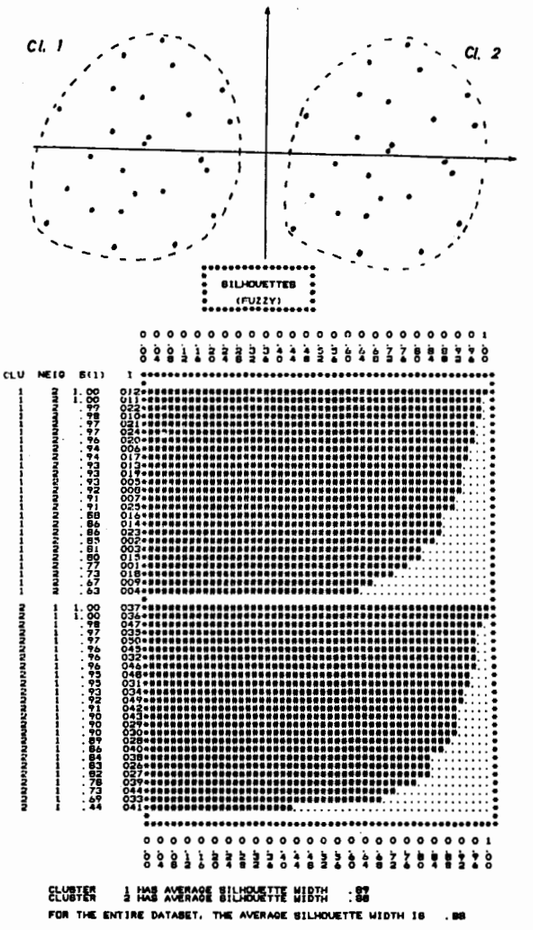

Fig.7: Fuzzy silhouettes of basic model: 2 clusters.

5° A two-dimensional plot with fuzzy membership functions

As in the case of the original silhouette, it is also possible to unfold the fuzzy membership function in a two-dimensional plot. Compared to section 3, there are two main differences:

1° due to relation (9) all points will lie above the 45° line;

2° relation (7) induces a series of constraints which were absent in the hard approach. As we will see, these depend on the number of clusters that is considered;

a) for 2 clusters, relation (7) becomes

$$u^{\circ}(i)+u^{\circ\circ}(i) = 1$$

and through (5) and (6) we find

$$a(i)+b(i) = 1. \qquad (10)$$

This relation means that all objects in a two-cluster system will be represented on the straight line going from (1,0) to (0,1) (see Fig.8).

b) for 3 clusters, relation (7) becomes

$$u^{\circ}(i) + u^{\circ\circ}(i) + u(t,i) = 1$$

or $u^{\circ}(i) + u^{\circ\circ}(i) \leq 1$

which through (5) and (6) gives

$$a(i) + b(i) \geq 1 \tag{11}$$

and as $u^{\circ\circ}(i) \geq u(t,i)$ through (6) we also have

$$u^{\circ}(i) + 2u^{\circ\circ}(i) \geq 1. \tag{12}$$

Using (5) and (6) this yields

$$1-a(i) + 2(1-b(i)) \geq 1$$

so

$$b(i) \leq 1 - \tfrac{1}{2}\, a(i). \tag{13}$$

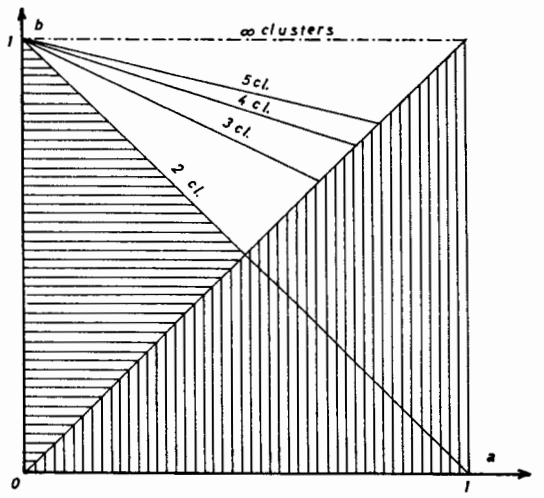

Fig.8: Two-dimensional plot with feasibility regions as function of number of fuzzy clusters.

Relations (11) and (12) force all objects in a three-cluster configuration to remain between two straight lines starting from the y-axis at the value b(i)=1 and with slopes -1 and $-\frac{1}{2}$ (see Fig.8).

c) for k clusters, relation (11) is still valid whereas relation (12) becomes

$$u^{*}(i) + (k-1)\, u^{**}(i) \geq 1 \qquad (14)$$

which upon consideration of (5) and (6) becomes

$$1 - a(i) + (k-1)(1 - b(i)) \geq 1$$

so

$$b(i) \leq 1 - 1/(k-1)\, a(i). \qquad (15)$$

Hence the lower and right hand feasibility limits (11) and (9) remain unchanged whatever the number of clusters; the upper limit starts from the point on the b(i) axis with value 1 and has a negative slope proportional to 1/(k-1) (see Fig.8). This upper limit coincides with the lower limit in the case of only two clusters (k=2) and tends to an horizontal line for an infinite number of clusters (k=OO). It can further be observed that whenever points are represented on the lower limit, i.e. when the sum of a(i) and b(i) is equal to one, these objects have zero membership to all clusters but the principal one and the first neighbour; points represented on the upper limit line corresponding to the number of clusters, indicate that equation (14) has to be considered with an equality sign and hence that

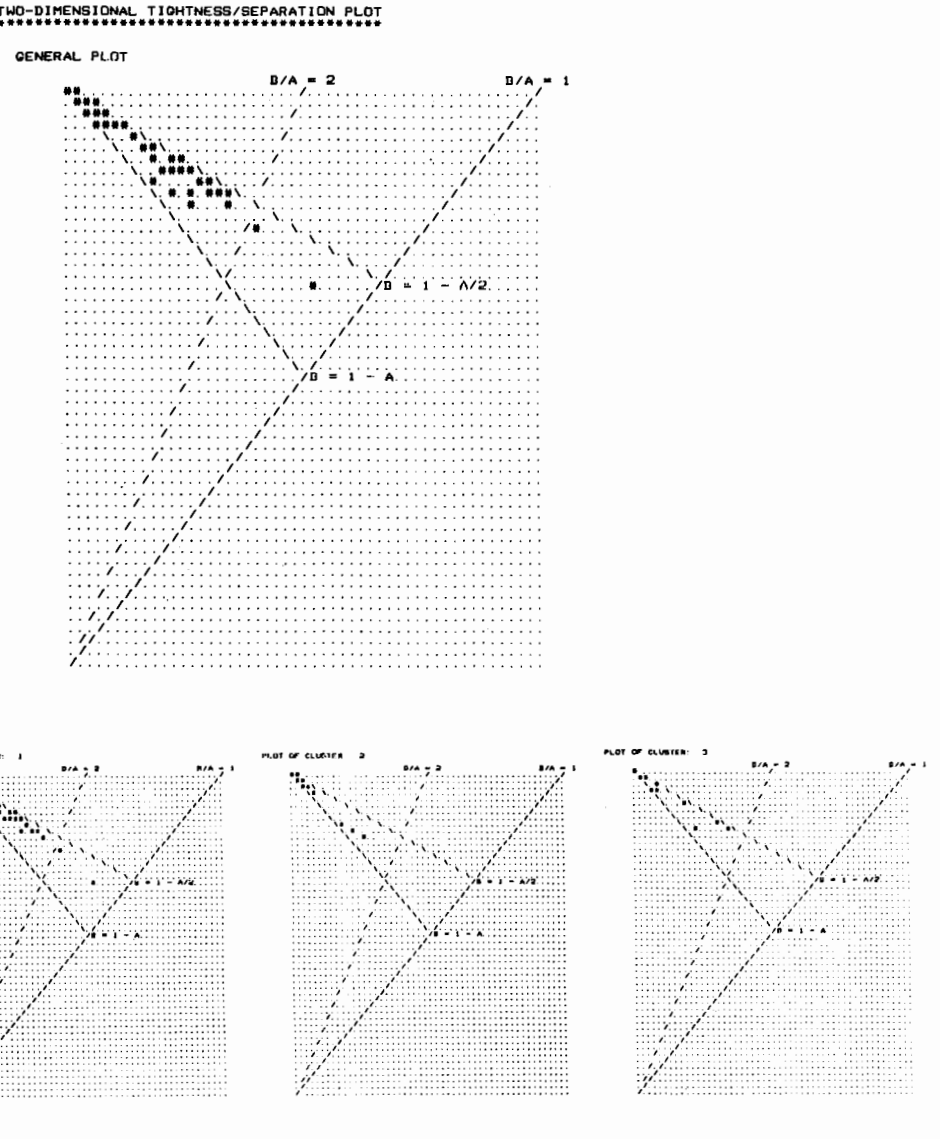

Fig.9: Ruspini's data: two-dimensional fuzzy representation of 3 clusters.

the corresponding object, apart from its membership to its principal cluster, has an equal membership to all the other clusters.

An example is provided by the Ruspini data [8], which contain four rather well-separated clusters. A partitioning into three fuzzy clusters shows two well-characterized clusters and a third one that is not so tight (Fig. 9). The partition in four clusters gives an improved image for all clusters, confirming the existence of four "natural clusters" (Fig.10).

6° Conclusions

Graphical representations are very useful to get a global impression of a clustering. It was shown how silhouettes could be extended to a two-dimensional plot, providing some new information such as a distinction between bridging objects and outliers.

A similar plot can be constructed from fuzzy membership functions. There all points remain within a triangle, of which only the upper boundary is a function of the number of clusters. Moreover, the position of each object within this triangle tells a lot about the clustering characteristics.

As seen from the examples, the above graphs can even be drawn with a plain line printer. This allows the implementation of these graphical representations in almost any computer environment.

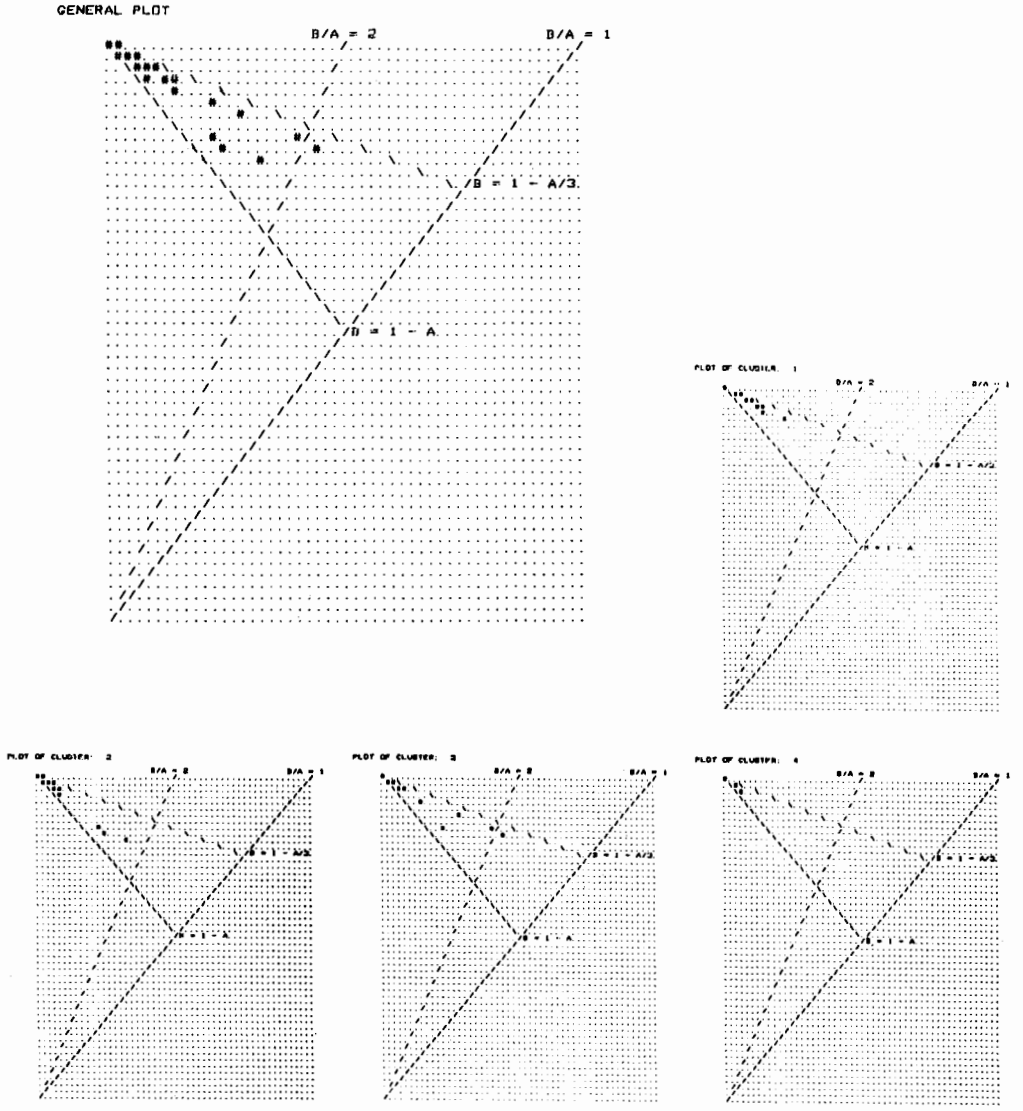

Fig.10: Ruspini's data: two-dimensional fuzzy representation of 4 clusters.

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