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LOGICALLY CONSISTENT MARKET SHARE MODELS REVISITED

Marcel WEVERBERGH

Director, Centre for Managerial Economics and Econometrics (C.M.E.E.)
UFSIA, University of Antwerp.

Philippe NAERT

Professor, UFSIA, University of Antwerp and Director of the European Institute for Advanced Studies in Management (E.I.A.S.M.), Brussels.

Alain BULTEZ

Professor, Faculté Universitaire Catholique de Mons and E.I.A.S.M.

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1. INTRODUCTION

In an article published recently in the Revue Belge de Statistique, d'Informatique et de Recherche Opérationnelle, Paelinck and Tack [23] reviewed extensively some of the issues related with the estimation of logically consistent econometric models[1]. More

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[1] That is, models specified in such a way that logical constraints defining the range of variation of the dependent variable be automatically satisfied by the model-predicted values. For an interesting discussion of the concept of logical consistency, see Koehler and Wildt [10].

pecifically, the authors focused on "spatial interaction models", .e. "models that distribute certain activities (working, purchasing, ravelling) among the parts of a study area ". The flows that such odels try to explain have to add up to the total flow in the system nd as a result, either the specification, or the parameter-estimators, f these models have to be properly constrained. For example, in the arketing context, competing brands' shares sum to one therefore, for ogical-consistency sake, model-predicted shares must also add up to 00 %.

Naert and Bultez [18,19] were the first ones - in the marketing context - to point out the necessity of resorting to consistent sum - constrained specifications. Later on, Nakanishi and Cooper [21,22], Bultez and Naert [8,9] and then Bultez [6,7] discussed at length, ordinary and generalized least-square as well as maximum-likelihood estimation procedures of the parameters of such market-share (attraction) functions. In parallel to the development of estimation methods, a controversy opposed Bultez and Naert, joined by Weverbergh [28], to the tenants of linear market-share equations (Beckwith [4], e.g.)[1]. Most unfortunately, a major part of our argumentation has remained unpublished so far. This is the reason why we take this opportunity to clarify and extend our previous work dealing with constraints to be imposed on the parameters of linear models, so that they be logically consistent.

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[1] Partly rejected by Bultez and Naert, due to lack of robustness (or consistency), a question examined hereafter.

Although illustrated, by examples from marketing, we believe our conclusions readily apply to various types of interaction models, especially to the "attraction-constrained" model category, as defined by Paelinck and Tack [23,p.20].

In the abstract of the latest article published on this topic (in the marketing literature, at least), McGuire and Weiss [16,p.296] proposed to "discuss, explain, and position the more notable literature on logically consistent market share demand models". It is our feeling that they fell somewhat short of their objectives in the sense that a number of contributions, were omitted from their analysis. First, we comment on these, in order to put the McGuire and Weiss paper in a proper perspective.

Next, McGuire and Weiss [16,p.296] correctly observed that there was an error of omission in the proof of the Naert-Bultez theorem on the necessary and sufficient conditions for a linear model to predict sum constrained dependent variables [18,p.339]. Indeed, Naert-Bultez failed to regard "homogeneous" variables as possible explanatory variables. Hence they neglected the constraints implied by such a type of variables. In their proof, however, there was an implicit assumption which ruled out the possibility of homogeneous variables. Therefore, in this article, we want to clarify this issue and examine in a more general way, the relation that exists between the number of constraints on the parameters, and the number of constraints on the explanatory variables.

Finally, we point to some misinterpretations that were given to our previous work on the subject.

2. IMPLICATIONS OF THE LOGICAL CONSISTENCY REQUIREMENT

Observed market shares have, by definition, values between zero and one, and when summing across brands a value of one is obtained. In market-share response functions, individual observations on the dependent variables are therefore restricted in range, and at the same time theysatisfy a sum constraint. Logical consistency of the model specification or structure then means that model predictions satisfy these same constraints[1]. The first question N-B raised in [18] concerned the implications of logical consistency for such frequently used market-share response functions, as the linear and multiplicative ones. It should be obvious that the latter specifications do not satisfy these constraints as such. They do not necessarily result in predicted values between zero and one, nor do market shares sum to one.

As far as the linear model is concerned it has been shown by McGuire, Farley, Lucas and Ring [15] and by Naert and Bultez [18] that the sum constraint (not the range constraint) can be satisfied if and

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[1] Logical consistency also applies to other types of dependent variables, such as, brand sales which should sum to product class sales. Also sum-constrained models are not limited to marketing problems but apply to such areas as international trade (import-export flows), migration, input-output analysis, and demand analysis. See B-N [9] and Paelinck and Tack [23].

only if a particular set of constraints on the explanatory variables and on the parameters is satisfied. Naert and Bultez [18,p.336] have argued that while it is possible to derive mathematical restrictions on the parameters, these will not necessarily be meaningful. For example, for the sake of logical consistency one may have to impose equality constraints on the parameters of a given marketing instrument across brands. Consider, e.g., the following market-share equation :

$$m_{it} = \alpha_i + \beta_i \, m_{i,t-1} + \gamma_i \, a^*_{it} \quad ,$$

for $i = 1, 2, \ldots, n$;
with m_{it} representing brand i's market-share in period t, and a^*_{it}, brand i's share of the industry overall advertising effort, during the same period (i.e. $a^*_{it} = a_{it} / \sum_{i=1}^{n} a_{it}$, a_{it} standing for brand i's advertising budget). To get predicted shares which add up to unity, N-B showed that the following constraints had to be imposed on the parameters[1] :

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[1] Sufficiency of these conditions is easily established, since by definition :

$$\sum_{i=1}^{n} m_{it} = \sum_{i=1}^{n} m_{i,t-1} = \sum_{i=1}^{n} a^*_{it} = 1$$

and thus

$$\sum_{i=1}^{n} m_{it} = \sum_{i=1}^{n} \alpha_i + \beta + \gamma .$$

Necessity can be demonstrated by reference to the theorem presented in the Appendix and discussed in section 3 (example 1).

$$\beta_i = \beta \text{ and } \gamma_i = \gamma, \text{ for all } i$$

and

$$\sum_i \alpha_i = 1 - \beta - \gamma .$$

Few practitioners would find these acceptable specification characteristics. The notion that model structure, including constraints to make it logically consistent in a mathematical sense, should have economic meaning and appeal is also lacking in Beckwith's [4] counterexample to N-B theorem[1]. McGuire and Weiss do pay attention to this point when discussing Beckwith's example, but it can hardly be stressed enough. Since the case of the McGuire and Weiss paper deals with the linear model we will come back to their analysis more extensively in the following section.

For the multiplicative model, such as the following (double-log) variant of the above linear specification,

$$m_{it} = \alpha_i \, m_{i,t-1}^{\beta_i} \, a_{it}^{*\gamma_i} ,$$

which is perhaps the most popular market-share specification, constraints on parameters and explanatory variables (except for trivial and totally uninteresting ones, e.g., $\beta_i = 0$, $\gamma_i = 1$ and $\alpha_i = 1$) cannot be derived in order to satisfy the sum constraint on the dependent variable.

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[1] Developed hereafter, in section 3.

Because of the shortcomings of both the linear and multiplicative models when logical consistency is required, Naert and Bultez concluded that other, probably more complex, specifications should be used, which inherently satisfy both the range and sum constraints[1]. Along similar lines Little [13,14] has argued that models should be robust, that is, "the user should not be able to push it to extremes that produce absurd results" [14,p.630]. In the examples in both [13] and [14], Little has been more concerned with the range constraint than with the sum constraint. Indeed, assuming symmetric specifications for all brands, Little's specification violates the sum constraint. Once again, this need not lead us to reject the specification, since intented use will be an important determinant of whether or not a model is acceptable and useful[2].

An interesting class of models; having a structure satisfying both range and sum constraints, consists of those generally known as <u>attraction models</u>. The attraction of a brand depends on its marketing mix. Let α_{it} be the attraction, or in fact the attraction function,

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[1] This by no means implies that linear or multiplicative models should never be used again. Much depends on what the model is intended to be used for. For a more extensive discussion of this point see N-B [19] and Naert and Leeflang [20, chapter 6].

[2] In this context it is interesting to refer to Lilien's concept of Model Relativism [12].

of brand i in period t. Market share attraction models are defined as,

$$m_{it} = \frac{\alpha_{it}}{\Sigma_{i=1}^{n} \alpha_{it}} , \qquad (1)$$

where n is the number of brands competing on the market. If α_{it} is specified to be nonnegative, the attraction model has the desirable characteristics of both satisfving the range constraint ($0 \leqslant m_{it} \leqslant 1$), and the sum constraint ($\Sigma_i m_{it} = 1$). Bell, Keeney, and Little [5] have demonstrated that the following axioms necessarily lead to such a market share attraction model :

a) The attraction for each brand is nonnegative, that is, $\alpha_{it} \geqslant 0$ for i = 1, ..., n, and t = 1, ..., T, and total attraction exerted on the market is positive, $\Sigma_i \alpha_{it} > 0$, t = 1, ..., T.
b) No attraction implies zero market share.
c) Brands with equal attraction have identical market shares.
d) If the attraction of a brand changes by a given amount, market share of any of the other brands is affected equally, no matter which brand's attraction has changed.

Thus the attraction model is not just a model that by chance satisfies the market-share range and sum constraints, but it is a model structure which logically follows from a number of plausible axioms. At first sight, one might feel somewhat uncomfortable about axiom d. However, it does, not imply that a change of δ in, e.g., the advertising expenditures of brand c, or a change of δ in those of brand b, would have the same effect on the market share of brand i (c, b, and i are different brands). This can be made clear as follows. The attractions will in general be functions of the marketing instruments[1]. For example,

$$\alpha_{ct} = f_c(x_{c1t}, x_{c2t}, \ldots, x_{ckt}), \text{ and } \alpha_{bt} = f_b(x_{b1t}, x_{b2t}, \ldots, x_{bkt}) ,$$

where x_{bst} is the value of variable s, for brand b, in period t. Changing the advertising expenditures level of brand c, say x_{c2t}, by δ will in general have a different effect on m_{it} than changing x_{b2t} by the same amount. This results from the possible asymmetry in the attraction functions, such as, differences in response parameters across brands. In addition the attraction functions may be nonlinear. We should, however, not conclude that axiom d is unrealistic, since it deals with equal changes in the attractions α_{ct} and α_{bt}, and not in the components of the attraction functions.

The problem of asymmetry and nonlinearities in relation to the Bell-Keeney-Little theorem has been examined by Barnett [1]. His elaboration of the theorem is based primarily on his finding that axiom c is not essential to their result.

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[1] There could also be other determining variables, such as, disposable income.

Equation (1) represents the overall structure. The attraction function itself remains to be specified. Nakanishi [21] proposed the following attraction model,

$$m_{it} = \frac{\alpha_{io} \Pi_{j=1}^{k} x_{ijt}^{\beta_j}}{\Sigma_{i=1}^{n} [\alpha_{io} \Pi_{j=1}^{k} x_{ijt}^{\beta_j}]} \tag{2}$$

where, x_{ijt} is the value of variable j, for brand i in period t, and the α_{io} and β_j, the model parameters. While similar formulations had been used before by, e.g., Kuehn, McGuire, and Weiss [11], Nakanishi was probably the first one to realize that (2) is not intrinsically nonlinear as had previously been believed. Nakanishi suggested a, be it non trivial, tranformation of (2), making it linear in the parameters[1]

We should observe that (2) does not contain an error term. In his later work with Cooper [22], however, Nakanishi explicitly considered a disturbance term as a multiplicative component in each of the attraction functions, following a suggestion by Bultez and Naert [8]. Nakanishi and Cooper developed a generalized least-square procedure for the case where in addition to the presence of the disturbance term, the observations on the dependent variable are sample data and are thus subject to sampling error.

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[1] Reported by Paelinck and Tack [23,pp.41-42].

The logical next step in going from (2) is to allow response parameters to vary across brands. This extension together with the explicitation of the disturbance terms, ε_{it}, in the attraction functions leads to,

$$m_{it} = \frac{\alpha_{i0} \Pi_{j=1}^{k} x_{ijt}^{\beta_{ij}} \exp(\varepsilon_{it})}{\Sigma_{i=1}^{n} [\alpha_{i0} \Pi_{j=1}^{k} x_{ijt}^{\beta_{ij}} \exp(\varepsilon_{it})]} . \qquad (3)$$

This model can again be linearized following the procedure used by Nakanishi to transform (1). Bultez and Naert derived the properties of the error term of (3) in [8], and of a somewhat more general vari in [9]. They demonstrated how, after linearization, equation (3) can be estimated by the generalized least-square procedure proposed by McGuire, Farley, Lucas and Ring in [15]. They also showed how equati (3) relates to Theil's multinominal extension of the linear logit model [26]. Simply taking the ratio of m_{it} and m_{bt} $(b \neq i)$, we obtai

$$\frac{m_{it}}{m_{bt}} = \frac{\alpha_{i0}}{\alpha_{b0}} \Pi_{j=1}^{k} (x_{ijt})^{\beta_{ij}} (x_{bjt})^{-\beta_{bj}} \exp(\varepsilon_{it} - \varepsilon_{bt}), \qquad (4)$$

which fits Theil's definition. This consideration led B-N to propose an alternative to Nakanishi's linearization procedure, since (4) becomes linear upon taking logarithms[1]. It should also be clear that

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1 The relation between these and other linearization procedures for attraction models has been studied in details by Bultez in [6,pp.2 227] and [7].

(4) can be particularized by allowing parameters to vary across brands for some variables, and not for others.

We were pleased to learn from a reference in [16] that others had independently been working along similar lines [17].

3. *THE RELATION BETWEEN CONSTRAINTS ON PARAMETERS AND ON EXPLANATORY VARIABLES*

From the discussion of N-B theorem by Beckwith, as well as by McGuire and Weiss, one somehow gets the feeling that when dealing with sum-constrained linear models, there must be a relation between the constraints on the explanatory variables and those on the parameters. We will try to clearly show this relationship below.

Consider the following general linear model,

$$y_i = \alpha_i u_T + X_i\beta_i + \varepsilon_i \quad , \text{ for } i = 1, \ldots, n, \qquad (5)$$

where, y_i is a T x 1 vector of observations on the dependent variable related to cross-section i (e.g., brand i) ;

X_i is a nonstochastic T x k matrix of values taken by the explanatory variables, related to cross-section i ;

ε_i is the corresponding T x 1 vector of disturbance terms ;

β_i is a k x 1 vector of parameters ;

α_i is the constant term ;

u_T is a sum vector of order T, i.e. $u_T' = [1, 1, \ldots, 1]$.

We will assume T to be at least equal to (n.k+1), a condition which will be justified later on. The dependent variables satisfy the following sum constraint,

$$\Sigma_{i=1}^{n} y_i = r = r^{*} u_T, \qquad (6$$

where r^{*} is a scalar[1]. The system of T.n equations (5) can be summarized in matrix notation as,

$$y = [I_n \otimes u_T] \alpha + X \beta + \varepsilon, \qquad (;$$

where, $y' = [y_1', y_2', \ldots, y_n']$,

I_n is an identity matrix of order n,

$\otimes$ represents a Kronecker product,

$\alpha' = [\alpha_1, \alpha_2, \ldots, \alpha_n]$

$$X = \begin{bmatrix} X_1 & 0 \ldots & 0 \\ 0 & X_2 \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \ldots & X_n \end{bmatrix} ,$$

$\beta' = [\beta_1', \beta_2', \ldots, \beta_n']$, and

$\varepsilon' = [\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_n']$.

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1 In general the elements of r could be time dependent. Here we assume $r_t = r^{*}$, for all t. The case with variable r_t will be discussed briefly at the end of this section.

The sum constraint can be written as,

$$[u_n' \otimes I_T]\ y = r^{*}\ u_T\ . \qquad (8)$$

Premultiplying both sides of (7) by $[u_n' \otimes I_T]$, we obtain,

$$r^{*}\ u_T = [u_n' \otimes I_T][I_n \otimes u_T]\alpha + [u_n' \otimes I_T]X\ \beta + [u_n' \otimes I_T]\varepsilon. \qquad (9)$$

Since the left-hand side is nonstochastic, so must be the right-hand side. This implies that $[u_n' \otimes I_T]\varepsilon$ should be nonstochastic and thus identically equal to its expectation, that is,

$$[u_n' \otimes I_T]\varepsilon = 0\ .$$

Using this result and rearranging terms, we get,

$$[u_n' \otimes I_T]X\ \beta + u_T(u_n'\ \alpha - r^{*}) = 0\ ,$$

which can also be written as,

$$Z \begin{bmatrix} \beta \\ u_n'\ \alpha - r^{*} \end{bmatrix} = 0\ , \qquad (10)$$

where Z is a T x (n.k+1) matrix,

$$Z = [X_1,\ X_2,\ \ldots,\ X_n,\ u_T]\ . \qquad (11)$$

Defining $\left[\beta^{+}\right]' = \left[\beta', [u_n' \alpha - r^{**}]\right]$, it then follows that the vector β^{+} must be a solution to the system of homogeneous linear equations,

$$Z\ a = 0\ , \qquad (12)$$

where a is a vector of dimension (n.k+1)x1.

It can easily be shown that the number of independent constraints on the elements of the vector β^{+} is equal to the rank of Z. The constraints themselves are obtained as a set of vectors forming a basis for the null space of Z. For a demonstration, refer to the Appendix[1].

Two examples will be used to clarify the implications of the theorem.

Example 1 :

Consider three brands whose market shares are affected by two variables, that is,

$$y_i = \alpha_i u_T + X_{i1} \beta_{i1} + X_{i2} \beta_{i2} + \varepsilon_i, \text{ for } i = 1, 2, 3, \qquad (13)$$

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[1] A completely analogous result could be obtained for the case without constant term, or for a general X matrix (with or without constant term, or for the number of variables varying across brands). For a more general discussion we refer to Weverbergh [28], on which the theoretical part of this section is based. Further extensions can also be found in Koehler and Wildt [10].

with

$$\Sigma_{i=1}^{3} y_i = u_T \ , \qquad (14)$$

that is, market shares should sum to one ($r^{*} = 1$). Let the variables also be sum-constrained as follows,

$$\Sigma_{i=1}^{3} X_{i1} = u_T \ , \qquad (15)$$

and

$$\Sigma_{i=1}^{3} X_{i2} = u_T \ . \qquad (16)$$

For example, X_{11}, X_{21}, and X_{31} could be the observation vectors for the advertising shares of the three brands in the market (a^{*}_{it}), and X_{12}, X_{22}, and X_{32}, the observation vectors of lagged market shares ($m_{i,t-1}$).

The Z matrix defined in (11) then becomes,

$$Z = [X_{11}, X_{12} \ \vdots \ X_{21}, X_{22} \ \vdots \ X_{31}, X_{32} \ \vdots \ u_T] \ . \qquad (17)$$

From the constraints (15) and (16), it follows that Z will have a rank at most equal to five. For convenience of exposition we will assume that it is exactly equal to five[1]. From the theorem, there must then be five independent constraints on the elements of the vector $[\beta^+]'$, that is, on the vector

$$[\beta_{11}, \beta_{12} \mid \beta_{21}, \beta_{22} \mid \beta_{31}, \beta_{32} \mid (\alpha_1 + \alpha_2 + \alpha_3 - 1)] \ . \qquad (18)$$

Given (15) and (16) it is easily seen that the columns of the matrix Λ below form a basis for the null space of Z,

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} ,$$

$$\text{for : } Z\,\Lambda = \left[\sum_{i=1}^{3} X_{i1} - u_T \;\middle|\; \sum_{i=1}^{3} X_{i2} - u_T \right] = [0,0]$$

Since the transpose of (18) must be a linear combination of the columns of Λ, that is, $\beta^+ = \Lambda\,\lambda$, with $\lambda' = [\lambda_1, \lambda_2]$, we must have

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1 If we assume the number of rows (T) to exceed the number of columns (n.k+1 = 7) the rank will in most realistic cases, be five. It could be less if, for example, the ratio of advertising shares of two of the brands were constant in all time periods, an event which is highly unlikely.

$$
\begin{bmatrix} \beta_{11} \\ \beta_{12} \\ \hdashline \beta_{21} \\ \beta_{22} \\ \hdashline \beta_{31} \\ \beta_{32} \\ \hdashline \alpha_1+\alpha_2+\alpha_3-1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \hdashline \lambda_1 \\ \lambda_2 \\ \hdashline \lambda_1 \\ \lambda_2 \\ \hdashline -\lambda_1-\lambda_2 \end{bmatrix}
$$

From the above, the following independent restrictions on the parameters can be derived :

- Two restrictions on the β_{i1} 's, namely $\beta_{11} = \beta_{21}$, and $\beta_{11} = \beta_{31}$, and thus,

$$\beta_{11} = \beta_{21} = \beta_{31} = \lambda_1 . \tag{19}$$

- Two restrictions on the β_{i2} 's, namely $\beta_{12} = \beta_{22}$, and $\beta_{12} = \beta_{32}$, and thus,

$$\beta_{12} = \beta_{22} = \beta_{32} = \lambda_2 . \tag{20}$$

- One constraint relating the parameters and the sum-constraint,

$$\sum_{i=1}^{3} \alpha_i + \lambda_1 + \lambda_2 - 1 = 0. \tag{21}$$

After substituting out the restrictions on the β_{ij} 's, the λ_j's become the unknown coefficients[1] in the McGuire, Farley, Lucas and Ring model [15]. Compare with the first explicit example given in section 2, where : $\beta_i = \beta = \lambda_1$ and $\gamma_i = \gamma = \lambda_2$.

Example 2 :

Equations (13), (14), and (15) remain unchanged, but equation (16) is replaced by,

$$X_{12} = X_{22} = X_{32} ,$$

that is, the observation vector for the second variable is the same for each brand, and is therefore called a homogeneous variable[2].

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1 The constant terms in our model are called brand dummies in their's.

2 More generally, homogeneous variables mean, $b_1X_{12} = b_2X_{22} = b_3X_{32}$, where the b_i 's are scalars. A typical example of a homogeneous variable (with $b_1 = b_2 = b_3$) is disposable income. In a brand sales model, the explanatory role played by such a variable may be easily understood, since product-class sales may be income elastic. In a market share model, however, its presence among regressors can hardly be justified, for having disposable income as a determinant of market shares would imply that the various brands' sales could be differently affected by its level. Hence it would indicate a rather substantial product differentiation between brands and in such a case of monopolistic competition, a market-share model (designed for oligopolistic structures) would be inappropriate. Let us note furthermore that a market-share specification is often chosen precisely, to eliminate the influence of environmental variables (disposable income, weather conditions, ...) which are assumed to affect all brands in the same way.

It will be clear to the reader that Z now has a rank of at most four. As in the first example, we assume the rank to be exactly equal to four. Thus there must now be four constraints on the parameters, and it is easily verified that the columns of the matrix Λ below form a basis for the null space of Z,

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} .$$

$$\text{for } Z\Lambda = \left[\sum_{i=1}^{3} x_{i1} - u_T \;\vdots\; x_{12} - x_{22} \;\vdots\; x_{12} - x_{32} \right] = [0,0,0] ;$$

and thus we must have,

$$\beta^{+} = \Lambda \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 + \lambda_3 \\ \hline \lambda_1 \\ -\lambda_2 \\ \hline \lambda_1 \\ -\lambda_3 \\ \hline -\lambda_1 \end{bmatrix}$$

which implies,

$$\beta_{11} = \lambda_1,\ \beta_{21} = \lambda_1,\ \beta_{31} = \lambda_1\ ,$$
$$\beta_{12} = \lambda_2 + \lambda_3,\ \beta_{22} = -\lambda_2,\ \beta_{32} = -\lambda_3\ ,$$
$$\alpha_1 + \alpha_2 + \alpha_3 - 1 = -\lambda_1\ .$$

It follows that,

$$\beta_{11} = \beta_{21} = \beta_{31} = \lambda_1\ ,$$
$$\beta_{12} + \beta_{22} + \beta_{32} = 0, \text{ and}$$
$$\Sigma_i\alpha_i + \lambda_1 - 1 = 0.$$

Discussion :

The results of the first example are the same as those obtained by applying the Naert-Bultez theorem. The second example illustrates how the conditions have to be changed if some or all of the variables are homogeneous instead of sum-constrained.

Schmalensee has also derived necessary and sufficient conditions which parameters of linear sum-constrained models must satisfy in order to be logically consistent [25,p.109-11]. He obtained the same results as those derived in [18], and it might be of interest to discuss his analysis. In a first theorem, he demonstrates that if for a given explanatory variable, the n corresponding observation vectors are linearly independent, then either, the corresponding coefficients must be zero, or the variable vectors must be sum-constrained. From this he concludes that "for all practical purposes", variables which are not sum-constrained cannot enter the model [25,p.111]. In a second theorem, he then derives necessary and sufficient conditions for logical consistency, assuming linear independence for the n observations vectors corresponding to each of the explanatory variables. This assumption in fact excludes homogeneous variables from entering the model. In his doctoral dissertation this point is made more explicit, when in commenting his first theorem, he states : "Other results of this kind are possible, of course. For instance, if all components of X_{ij} are equal for all i, it is clear that the sum of the corresponding coefficients must be zero" [24,p.49, and slightly adapted to fit our notation].

Beckwith's counterexample

In [4], Beckwith has presented a counterexample to the Naert-Bultez theorem, i.e. the following specification,

$$y_i = X_i \beta_i + \varepsilon_i, \text{ with } \beta_i' = [\beta_{i1}, \beta_{i2}, \ldots, \beta_{ik}] ,$$

for all i but the last one, and

$$y_n = r^* u_T - \sum_{i=1}^{n-1} X_i \beta_i + \varepsilon_n .$$

The equation corresponding to the last group of observations is thus defined in an ad hoc manner, to meet the sum constraint.

Transforming his case to three brands and two variables, as in our previous examples, and not imposing any prior constraints on the parameters, we can rewrite it as

$$y_1 = X_{11}\beta_{11} + X_{12}\beta_{12} + \varepsilon_1 \qquad (22)$$

$$y_2 = X_{21}\beta_{21} + X_{22}\beta_{22} + \varepsilon_2 \qquad (23)$$

$$y_3 = \alpha_3 u_T + X^1_{31}\beta^1_{31} + X^1_{32}\beta^1_{32} + X^2_{31}\beta^2_{31} + X^2_{32}\beta^2_{32} + \varepsilon_3 \, . \qquad (24)$$

In Beckwith's example,

$$X^i_{3k} = -X_{ik} \text{ for } i = 1, 2 \text{ and } k = 1, 2,$$

that is, the variables are sum-constrained in the following way,

$$X^i_{3k} + X_{ik} = 0 \text{ for } i = 1, 2 \text{ and } k = 1, 2.$$

The dependant variables are also sum-constrained,

$$\Sigma^3_{i=1}\, y_i = r^* \, u_T .$$

Given the constraints on the explanatory variables, the rank of the Z matrix,

$$Z = [X_{11}, X_{12} \mid X_{21}, X_{22} \mid X^1_{31}, X^1_{32}, X^2_{31}, X^2_{32} \mid u_T]$$

will normally be equal to five. The columns of Λ below form a basis for the null space of Z,

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for $Z\,\Lambda = [X_{11} + X_{31}^{1} \mid X_{12} + X_{32}^{1} \mid X_{21} + X_{31}^{2} \mid X_{22} + X_{32}^{2}]$,
or, $Z\,\Lambda = [0,0,0,0]$, since : $X_{ik} + X_{3k}^{i} = 0$.
Since β^{+} must be a linear combination of the columns of Λ, the following constraints must be satisfied,

$$\begin{aligned} \beta_{11} &= \beta_{31}^{1} = \lambda_1 , \\ \beta_{12} &= \beta_{32}^{1} = \lambda_2 , \\ \beta_{21} &= \beta_{31}^{2} = \lambda_3 , \\ \beta_{22} &= \beta_{32}^{2} = \lambda_4 , \\ \sum_{i=1}^{3} \alpha_i &- r^{*} = 0. \end{aligned} \qquad \underline{\text{since}} : \Lambda \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ 0 \end{bmatrix}$$

Observing that (22) and (23) do not contain a constant term, that is, $\alpha_1 = \alpha_2 = 0$, the last expression reduces to

$$\alpha_3 = r^{*}.$$

Substituting these constraints into equation (24), we obtain the model specified by Beckwith. In so far as explanatory variables can be treated as sum-constrained, these same parameter constraints can also obtained by applying the Naert-Bultez theorem.

To do justice to Beckwith, however, we should say that his example could also be presented as one with homogeneous variables, by defining them as follows,

$$X^i_{3k} = X_{ik}, \text{ for } i = 1, 2 \text{ and } k = 1, 2.$$

The rank of Z remains, of course, unchanged. Applying the theorem presented in this paper we ultimately obtain,

$$\beta_{11} = \lambda_1, \ \beta^1_{31} = -\lambda_1, \text{ or } \beta_{11} + \beta^1_{31} = 0,$$

$$\beta_{12} = \lambda_2, \ \beta^1_{32} = -\lambda_2, \text{ or } \beta_{12} + \beta^1_{32} = 0,$$

$$\beta_{21} = \lambda_3, \ \beta^2_{31} = -\lambda_3, \text{ or } \beta_{21} + \beta^2_{31} = 0,$$

$$\beta_{22} = \lambda_4, \ \beta^2_{32} = -\lambda_4, \text{ or } \beta_{22} + \beta^2_{32} = 0, \text{ and}$$

$$\alpha_3 = r^* .$$

Thus we can regard Beckwith's example both as one with homogeneous (identical) or as one with sum-constrained (summing to zero) variables, because each time the homogeneity only involves two variables[1].

Abstracting from the fact that Beckwith's intention was to present a counterexample, a final comment should be made. It does not relate to the derivation of constraints, but is nevertheless very important. The fact that a model is logically consistent, does not mean that it has any economic meaning. Why should, for example, the specification of brand 3, in his case, be so different from that of the other brands ? Why should brand 3's market share be determined by variables relating to brands 1 and 2, but not to brand 3 itself ? A similar line of thought is discussed in McGuire and Weiss [16,p.301] .

It is our hope that the theoretical development presented at the beginning of this section may serve to spread some new light on logical consistency of linear models. The theorem derives its major interest from the fact that it allows for both homogeneous and sum-constrained explanatory variables, by concentrating on the relation that must exist between parameter and variable constraints.

Varying r_t

Up to now we have assumed that $r = r^* u_T$, that is $r_t = r^*$, for all t. Let us now explore the case of varying r_t. McGuire and Weiss have

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[1] When more than two variables are involved, regarding homogeneous variables as sum-constrained ones results in loss of degrees of freedom in the parameter vector.

argued that if the sum constraint is known for each time period, the problem is irrelevant. We do not quite agree. Suppose that brand sales equations were used instead of market share equations. Logical consistency would require brand sales to sum to industry or product-class sales, a time varying quantity. This does not seem irrelevant to us. The problem may perhaps be trivial, but that is another matter[1].

In [18] one of the conditions was $\Sigma_{i=1}^{n} x_{ijt} = c_{jt}$. According to McGuire and Weiss this is a definition, not a condition. It should be clear, however, that c_{jt} cannot take on any arbitrary value, since logical consistency conditions must be looked at simultaneously and not in isolation. Thus, excluding homogeneous variables, the c_{jt} should satisfy the constraint,

$$\Sigma_{i=1}^{n} \alpha_i + \Sigma_{j=1}^{k} \lambda_j c_{jt} = r_t , \tag{25}$$

as shown in [18]. It would then follow that for $r_t = r^*$, the explanatory variables must sum to a constant value, that is, $c_{jt} = c_j$. That is, with $k = 2$, $n = 3$, $c_1 = c_2 = 1$, $r^* = 1$, condition (25) reduces to (21).

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1 These constraints are comparable to the identities in macro economic simultaneous equation systems. For example, taking a very simplified version of such models, we have as a constraint that consumption plus savings must be equal to income, which varies over time. Nobody would consider these restrictions to be irrelevant.

In our previous work we also discuss logical consistency on a subset of brands. McGuire and Weiss criticize this by writing that only if a subset of brands can be treated as a market (independent of the deleted brands), sum constraints on that subset become relevant. Or to quote them, "... we would call the subsystem a complete system and normalize the market share (and explanatory variables where appropriate)..." [16,p.300]. This is exactly the conclusion we arrived at in [18]. We used a market studied by Beckwith ([2],[3]) to illustrate some of the implications of taking a subset of brands. Beckwith considered five brands, representing about ninety eight percent of the market. Two remaining brands served very specialized market segments. In his study, it was assumed that brands other than the five under study had no advertising expenditures [2,p.56], that is, they were normalized. Our analysis led us to conclude : "It would be better in this case to regard the market captured by the 5 brands as the total market, i.e., express the market shares of the individual brands as a percentage of r_t" [18,p.336]. Perhaps we did not sufficiently stress the implications of this conclusion.

4. *CONCLUSION*

In this paper we have derived logical consistency conditions by relating constraints on explanatory variables and on parameters, in linear sum-constrained models. This approach allows to simultaneously consider homogeneous and sum-constrained explanatory variables, thus eliminating the confusion, errors, and misinterpretations that have plagued the area.

While this and some previous papers have contributed to our understanding of logical consistency in linear models, we want to stress again as we did in [18], that the conditions are such as to indicate some major limitations of linear market share models. If logical consistency is desired, linear specifications are not particularly appropriate and other structures such as, for example, attraction models are to be recommended.

APPENDIX

Let, $Z = [X_1, X_2, \ldots, X_n, u_T]$, and

$$[\beta^+]' = [\beta_1', \beta_2', \ldots, \beta_n', (u_n' \alpha - r^*)] ,$$

where, X_i = a Txk_i observation matrix related to cross-section i (excluding the constant term),

u_T = a Tx1 sum vector,

β_i = a $k_i x1$ - vector of parameters relating to cross-section i,

α = the nx1 vector of constant terms,

r^* = the value the dependent variables sum to in each period.

Theorem :

The necessary and sufficient conditions relating constraints on parameters and on explanatory variables are given by,

$$\beta^+ = \Lambda \lambda , \qquad (A.1)$$

where, Λ is a $(\Sigma_i k_i + 1) x \upsilon$ matrix, whose columns form a basis for the null space of Z, and where λ is a $\upsilon x1$ vector of proportionality factors[1].

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[1] For related work see, for example, Theil [27,chapter 7].

Proof :

In the body of the paper (equation (10)) it has been shown that sum-constrained dependent variables require,

$$Z \, \beta^{+} = 0, \tag{A.2}$$

that is, the vector β^{+} must be a solution to the system of homogeneous linear equations,

$$Z \, a = 0. \tag{A.3}$$

The theorem states that Λ consists of υ linearly independent vectors satisfying (A.3). Since the columns of Λ form a basis for the null space of Z, the latter's rank must equal η, where $\eta = ((\Sigma_i k_i) + 1) - \upsilon$.

Sufficiency :

$\Lambda \, \lambda$ is a linear combination of vectors satisfying (A.3), and must therefore also satisfy (A.2).

Necessity :

Suppose there exists a solution $\beta^{+} = \beta^{*}$, such that, $\beta^{*} \neq \Lambda \, \lambda$. Since β^{*} is a solution to (A.3), that is, $Z \, \beta^{*} = 0$ it must belong to the null space of Z. Since, however, β^{*} cannot be written as a linear combination of the columns of Λ, it would follow that these do not form a basis for the null space of Z, which is in contradiction to the definition of Λ. The vector β^{*} can therefore not be a solution of $Z\ a = 0$, which completes the proof.

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