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Comparing due-date-based performance measures for queuing models

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Abstract

Queueing models provide a useful tool in designing manufacturing systems amongst others. An extensive survey on the subject is given in a recent review paper of Buzacott and Shanthikumar(1992). However, these models neglected the due-dates of orders being processed. In this work, we describe a simple method for modelling such due-date-based performance measures as tardiness or flow time. Next, we derive bounds and comparison properties of these measures.

Key-words: Queueing systems, Reliability, Manufacturing systems, Stochastic comparisons, Due-date, Performance measures, Ageing distributions.

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1 Introduction

Manufacturing environments are naturally characterized as queueing systems or networks, which provides a useful tool for designing purposes. Buzacott & Shanthikumar [2, 3] gave an extensive review and bibliography on design issues, using Queueing models, in various types of manufacturing systems such as flow lines, automatic transfer lines, jobshops, flexible machining systems, flexible assembly systems and multiple cell systems. The interest of that paper is to show how the structural properties recently derived for Queueing systems can be used effectively in the solution of certain design optimization problems, while the classical theory of queues is restricted to the derivation of (explicit or implicit) formulas for the performance measures of interest.

In the case of manufacturing environments, these performance measures are not necessarily those addressed by classical Queueing Theory. However, only a few works take into consideration queueing models which incorporate the due-dates of the orders being processed, and this aspect don't appear in the cited above review.

To the best of our knowledge, the notion of lead-time for an "urgency class" was first introduced by Jackson [11, 12] and many variants were introduced later as "Earliest start time first protocol", "Weighted-delay protocol" [7, 8, 15, 17]. Seidman & Smith [18], Tate [21] presented some techniques for modelling due-date in a classical Queueing framework and their applications to design optimization problems.

In this work, we study the comparability of the introduced due-date-based performance measures in the same framework. For this purpose, comparison methods and reliability concepts of "ageing" will play a central role. In the section 3 we introduce the notion of stochastic ordering and some reliability concepts of "ageing". In the sections 4 and 5 we study the comparability of the tardiness and the flow time respectively. In section 6, we provide bounds on these measures.

2 Due-date-based performance measures.

Let us consider a single-server $GI/G/1$ queue in which orders arrive at rate λ and are processed in the order of their arrival at rate μ. Denote by t_n the date of the nth arrival and τ_n his service time, $n \geq 1$. Let $A(x) = P(\xi_n \leq x)$, where ($\xi_n = t_n - t_{n-1}, t_0 = 0$) and $B(x) = P(\tau_n \leq x)$ be the probability distribution of the stationary sequences $\{\xi_n\}$ and $\{\tau_n\}$ respectively. We denote by ξ, τ the corresponding stationary random variables.

Let $W_n(x) = P(w_n \leq x)$ be the probability distribution for the waiting time of the nth order, and $V_n(x) = P(S_n \leq x)$ be the probability distribution for sojourn time of this nth order. The condition $\mu^{-1} = \mathbb{E}(\tau_n) < \lambda^{-1} = \mathbb{E}(\xi_n)$ insure that there exist steady-state probability distribution for the waiting time $W(x) = \lim_{n\to\infty} P(w_n \leq x)$, and the sojourn time $V(x) = \lim_{n\to\infty} P(S_n \leq x)$. We denote by w and S the corresponding stationary random variables.

Seidman & Smith [18] and Tate [21] defined the lead-time L_n of the nth order to be the difference between order's due-date and its arrival epoch and $\Lambda(x) = P(L_n \leq x)$, provided the lead-times of distinct orders are independent and identically distributed random variables. The first due-date-based performance measure of our queue is the tardiness. More precisely, the tardiness T_n of the nth order is by definition :

$$T_n = \max\{0, S_n - L_n\} = (S_n - L_n)_+,$$

with probability distribution

$$\Gamma_n(x) = P(T_n \leq x), \quad x \geq 0.$$

The flow time F_n of the nth order is defined as the amount of time the order is in the processing facility

$$F_n = \max\{S_n, L_n\},$$

and

$$\Phi_n(x) = P(F_n \leq x), \quad x \geq 0,$$

since an order may often be shipped until its due-date, which distinguishe it from the classical sojourn time.

In the following, we denote by T and F the corresponding stationary due-date-based performance measures, with respective distributions $\Gamma(x) = \lim_{n\to\infty} P(T_n \leq x)$ and $\Phi(x) = \lim_{n\to\infty} P(F_n \leq x)$.

Seidman & Smith [18] considered the problem of the optimal choice of lead-time for an arbitrary mixture of convex cost functions involving tardiness penalties, earliness penalties and penalties for quoting long lead-times. Tate [21] showed how the iterative, numerical optimization algorithm of Seidman & Smith may be replaced by an analytical solution (in the case of exponential service time) for a much broader class of composite due-date-based objectives. They also gave a characterization of the due-date behavior of $G/M/1$ queueing systems facing exogenous random lead-times, modelled as Gamma random variables.

In the rest of the paper, we give conditions under parametric distributions (arrivals, service times and lead-times) for which the due-date-based performance measures of two queueing models are comparable in the sense of some defined stochastic orderings. Such results are important since they yields bounds on due-date-based performance measures understudy. On the other hand, they gave qualitative estimation of the structural properties for the optimization criterion introduced in [18, 21].

3 Stochastic orders and Reliability concepts of "ageing"

By the ageing of a physical or biological system, we mean the phenomenon by which an older system has a shorter remaining lifetime, in some stochastic sense, than a newer or younger one. The concept of "ageing" is widely used in last decade for the quantitative and/or qualitative estimation of the characteristics asssociated to stochastic models. Many criteria of ageing have been developped in the literature, but we mention only a few of them we need in what follows.

First, recall that the random variable X with distribution function F is smaller than a random variable Y with distribution function G relatively to the partial ordering " $<$ ", if

$$\int_{\mathbb{R}} f(x)dF(x) \leq \int_{\mathbb{R}} f(x)dG(x), \tag{1}$$

holds for all real functions $f \in \mathcal{F}$, where $\mathcal{F}$ is a given family of real functions. We equivalently write $X < Y$ or $F < G$ [1, 19, 20]. If $\mathcal{F}$ is the class of all non-decreasing functions f, then " $<$ " is the usual stochastic ordering or ordering in distribution, denoted by " $\leq_d$ ". The increasing convex (concave) ordering " $\leq_{icx}$ $(\leq_{icv})$" is obtained when $\mathcal{F}$ is the class of all non-decreasing convex (concave) functions.

Note that the increasing convex (concave) ordering may be defined by using some reliability concepts of "ageing". More precisely, $X \leq_{icx} Y$ if and only if for all real x

$$\begin{aligned} \mathbb{E}(X-x)_+ &= \int_x^{\infty}(X-t)dF(t) = \int_x^{\infty}[1-F(t)]dt \\ &\leq \int_x^{\infty}[1-G(t)]dt = \mathbb{E}(Y-x)_+ \end{aligned}$$

provided these expectations (equivalently, integrals) are finite. Interpreting X and Y as the lifetime of two components, we say that X and Y are comparable in mean residual life if $X \leq_{icx} Y$. Similarly, $X \leq_{icv} Y$ means that X and Y are comparable in mean used or elapsed life if $\mathbb{E}(x - X)_+ \geq \mathbb{E}(x - Y)_+$ (all real x) provided these expectations exists.

Note the following properties of the orders " $\leq_d$ ", " $\leq_{icx}$ " and " $\leq_{icv}$ " :

Proposition 1 *Let X and Y be two non-negative random variables, and " $<$ " be one of the orders " $\leq_{icx}$ ", " $\leq_{icv}$ " or " $\leq_d$ ", then*

$$X < Y \quad \Rightarrow \quad \mathbb{E}(X^k) \leq \mathbb{E}(Y^k) \quad (k \geq 0)$$

Proof : See Stoyan [20], chap.1, corollary 1.2.2.a. and corollary 1.3.1.a., 1.4.1.a. (see also Shaked and Shantikumar [19]).

Proposition 2

(i) $X \leq_d Y$ *and* $\mathbb{E}(Y_+) < \infty \;\Rightarrow\; X \leq_{icx} Y.$

(ii) $X \leq_{icv} Y \;\Leftrightarrow\; -Y \leq_{icx} -X.$

(iii) *If* $\mathbb{E}(X) = \mathbb{E}(Y)$, *then* $X \leq_{icv} Y \;\Leftrightarrow\; Y \leq_{icx} X$.

Proof : See Stoyan [20], chap.1, (i)p.8 ; (ii)p.10 ; (iii)p.11 (see also Shaked & Shantikumar [19]).

Definition 1 *We say that a given order " $<$ " on (a subset of) the space* $\mathbf{D}$ *of distribution functions has the convolution property if, whenever $F_1 < F_2$ and F_1, F_2 and $G \in \mathbf{D}$, the convolutions $F_i \otimes G_i (i = 1, 2)$, defined by $(F_i \otimes G_i)(x) = \int_{-\infty}^{\infty} F_i(x - y) dG(y)$ are elements of $\mathbf{D}$ and $F_1 \otimes G < F_2 \otimes G$.*

Proposition 3 *The stochastic order " $\leq_d$ " and the increasing convex order " $\leq_{icx}$ " have the convolution property.*

Proof: See Stoyan [20] chap.1, p.5 and p.9 (see also Shaked Shantikumar [19]).

Let X be a positive random variable with distribution function F and denote by X_t, a generic random variable with distribution function given by $(x \geq 0, t \geq 0)$

$$\begin{aligned} \bar{F}_t(x) &= 1 - F_t(x) = \mathbf{P}(X_t > X) \\ &\equiv \mathbf{P}(X > x + t \mid X > t) = \bar{F}(x + t)/\bar{F}(t) \end{aligned}$$

assuming $F(t) < 1$ and $\bar{F}(t) = 1 - F(t)$.

In reliability theory [1], X is interpreted as the lifetime of a component and F its survival (or reliability) function. Thus, X_t is the residual lifetime of a component of age t.

1. The distribution F of the non-negative random variable X has an **Increasing Failure Rate** or is an **IFR** distribution if :

   (i) $$F_y \leq_d F_x \ (\ all \quad 0 \leq x < y < \infty \ ). \qquad (2)$$

   The following criteria are equivalent forms of (2):

   (ii) $[F(t+x) - F(t)]/[1 - F(t)]$ is non increasing in t for each fixed $x > 0$.

   (iii) $-\log[1 - F(t)]$ is convex in x.
   If F has a density function, then F is **IFR** if and only if the failure rate or hazard rate $\lambda(x) = f(x)/\bar{F}(x)$ is non increasing in x.

2. F is **IFRA** (Increasing Failure Rate Average) if $-\frac{1}{t}\log \bar{F}(t)$ is increasing in $t \geq 0$.

3. F is said to be a **New Better than Used (NBU)** distribution if and only if $-\log[\bar{F}(t)]$ is superadditive, i.e. if
   $-\log[\bar{F}(x+y)] \geq -\log[\bar{F}(x)] - \log[\bar{F}(y)],\ x, y \geq 0.$
   This is equivalent to the statement

   $$F_t \leq_d F(\ all \ 0 < t < \infty) \ or \ \mathbf{P}(X > x + t/X > t) \leq \mathbf{P}(X > x).$$

   The **NBU** concept may be interpreted as that a used item of any age has (stochastically) smaller remaining life than a new item has.

4. The distribution function F of the random variable X is **New Better than Used in Expectation** (**NBUE**) if $\mathbb{E}(X_t) \leq \mathbb{E}(X)$ $(0 < t < \infty)$, or equivalently

   $$\int_t^{\infty} \bar{F}(x)dx \leq \mu \bar{F}(t). \qquad (3)$$

5. The distribution F is **HNBUE (Harmonic New Better than Used in Expectation)** if and only if

   $$\int_t^{\infty} \bar{F}(x)dx \leq m^{-1}e^{-t/m} \qquad \text{for} \quad t \geq 0, \qquad (4)$$

   and $m = \mathbb{E}(X) = \int_0^{\infty} \bar{F}(t)dt$.

The **HNBUE** and **HNWUE** classes of distributions are given in [14]. If the inequalities are reversed, then we speak of Decreasing Failure Rate (**DFR**), New Worse than Used (**NWU**), (**NWUE**), (**HNWUE**), ...

Note that we have the following chain of implications:

$$Exp \Rightarrow IFR \Rightarrow IFRA \Rightarrow NBU \Rightarrow NBUE \Rightarrow HNBUE$$

The interested reader may find the details about other distributions of ageing in the monographs of Barlow Proshan [1] and Stoyan [20]. A more recent classification based on the notion of s-failure rate is given in Faguioli & Pellery [6] (see also appendix A and B at the end of this paper).

Note the important following property of the class of ageing distribution functions :

Proposition 4 *[1, 20]*

$$(i) \qquad F \text{ is } \textbf{NBUE(NWUE)} \Rightarrow F \leq_{icx} (\geq_{icx}) \, Exp(m^{-1}), \tag{5}$$

where $Exp(.)$ is standing for the exponential distribution $Exp(\lambda t) = 1 - e^{-\lambda t}$ $(0 \leq t < \infty)$

$$(ii) \qquad F \text{ is } \textbf{IFR} \Rightarrow \bar{F}(t) \geq \begin{cases} e^{-t/m} & \text{for } t \leq m, \\ 0 & \text{for } t \geq m, \end{cases}$$

proof: See Barlow & Proshan [1] (corollary 6.3, p.112) for (ii) and proposition 1.6.2 of Stoyan [20] for (i).

4 Comparing tardiness distributions.

We return now to our $GI/G/1$ queueing model which incorporate due-dates. The following theorem gives conditions under parametric distribution functions for which the tardiness of two $GI/G/1$ queueing systems are comparable.

Theorem 1 *Let $\sum_1$ and $\sum_2$ be two $GI/G/1$ queueing in steady-state. i.e. $\rho_i = \mathbb{E}(\tau_i)/\mathbb{E}(\xi_i) < 1$ $(i = 1, 2)$, with parametric probability distributions A_i (interarrival times), B_i (service times) and Λ_i (lead-times), i=1,2.*

$$\begin{aligned} \textit{If} \qquad & A_2 \leq_{icv} A_1, B_1 \leq_{icx} B_2 \textit{ and } \Lambda_2 \leq_{icv} \Lambda_1 & (6) \\ \textit{then} \qquad & \Gamma_1 \leq_{icx} \Gamma_2 & (7) \\ \textit{and} \quad & \mathbb{E}(T_1) \leq \mathbb{E}(T_2). & (8) \end{aligned}$$

Proof. It is well known [15] that the waiting time w_n of the nth customer satisfies the recursion formula,

$$w_{n+1} = \max(0, w_n + U_n) \equiv (w_n + U_n)_+, \quad (n = 0, 1, \ldots),$$

where $U_n = \tau_n - \xi_n$. Then, the distribution function W_n of w_n satisfies the integral equation of Lindley

$$W_{n+1}(t) = \int_0^\infty K(t-x)dW_n(x) = \int_{-\infty}^\infty W_n(t-x)dK(x) \;\; (t \geq 0),$$

where $K(t) = \mathbf{P}(U_n \leq t)$ is independent of n, since the sequences $\{\xi_n\}$ and $\{\tau_n\}$ are stationary independent sequences of i.i.d. random variables. On the other hand, since $A_2 \leq_{icx} A_1$ and $B_1 \leq_{icx} B_2$, then by theorem 5.2.1 of Stoyan [20], we have for the stationary waiting time distributions $W_1 \leq_{icx} W_2$ and for the mean waiting times $\mathbb{E}(W_1) \leq \mathbb{E}(W_2)$.

By section 2, the tardiness distribution function is a negative convolution, and

$$\begin{aligned} \Gamma_i(x) &= \mathbf{P}\{(S_i - L_i)_+ \leq x\} \\ &= \begin{cases} \int_0^\infty V_i(x+y)d\Lambda_i(y) & x \geq 0 \quad (i=1,2) \\ 0 & x < 0 \end{cases} \end{aligned} \tag{9}$$

Since the increasing convex ordering " $\leq_{icx}$ " has the convolution property (see proposition 3 §3)], and $S_i = w_i + \tau_i$ (i=1,2), then it follows that

$$W_1 \leq_{icx} W_2, \qquad \text{and} \qquad B_1 \leq_{icx} B_2 \ \Rightarrow\ V_1 \leq_{icx} V_2.$$

From this inequality and taking into account formula (9), we have that

$$V_1 \leq_{icx} V_2, \qquad \text{and} \qquad \Lambda_2 \leq_{icv} \Lambda_1 \ \Rightarrow \Gamma_1 \leq_{icx} \Gamma_2,$$

The first part of theorem 1 is prooved. The second part follows immediately from the proposition 1 §3.

Remarks.

1. The convex comparability for the stationary waiting time w also holds for the sequence $\{w_n\}$. More precisely, if $K_1 \leq_{icx} K_2$, and $W_{0,1} \leq_{icx} W_{0,2}$, then for all $n = 0, 1, \ldots, W_{n,1} \leq_{icx} W_{n,2}$. This property is called the external monotonicity of $GI/G/1$ [20]. The condition $K_1 \leq_{icx} K_2$ is sufficient to ensure that $W_1 \leq_{icx} W_2$, provided the corresponding W_1 and W_2 exists.

2. In the case of equal means $\mathbb{E}(\xi_1) = \mathbb{E}(\xi_2), \mathbb{E}(L_1) = \mathbb{E}(L_2)$, we have that

$$A_1 \leq_{icx} A_2 \ \Leftrightarrow\ A_2 \leq_{icv} A_1 \qquad \text{and} \qquad \Lambda_1 \leq_{icx} \Lambda_2 \ \Leftrightarrow\ \Lambda_2 \leq_{icv} \Lambda_1.$$

Consider now some particular cases.

4.1 Poisson arrivals.

For $M/GI/1$ queue,

$$A_i(x) = 1 - e^{-\lambda_i x}, \text{ we have } A_2 \leq_{icv} A_1 \ \Leftrightarrow\ \lambda_1 \leq \lambda_2,$$

and the assertion of theorem 1 becomes

$$\lambda_1 \leq \lambda_2\ ,\ \ B_1 \leq_{icx} B_2\ ,\ \ \Lambda_2 \leq_{icv} \Lambda_1 \ \ \Rightarrow \Gamma_1 \leq_{icx} \Gamma_2.$$

4.2 Exponential service times.

For $GI/M/1$ queue, $B_i(x) = 1 - e^{-\mu_i x}$,

$$A_2 \leq_{icv} A_1 \ , \ \ \mu_1 \geq \mu_2 \ , \ \ \Lambda_2 \leq_{icv} \Lambda_1 \ \ \Rightarrow \ \ \Gamma_1 \leq_{icx} \Gamma_2 .$$

4.3 Exponential lead-times.

If $L_i(x) = 1 - e^{-\nu_i x}$,

$$A_2 \leq_{icv} A_1 \ , \ \ B_1 \leq_{icx} B_2 \ , \ \ \nu_1 \leq \nu_2 \ \ \Rightarrow \ \ \Gamma_1 \leq_{icx} \Gamma_2 .$$

5 Comparing Flow time distributions.

In this section, we prove a comparability result for the flow time, similar to that of theorem 1.

Theorem 2 *Let Σ_1 and Σ_2 be two $GI/G/1$ queues in steady-state, with parametric distributions A_i, B_i and Λ_i (i=1,2).*

$$\text{(i)} \quad \text{If} \quad K_1 \leq_d K_2, B_1 \leq_d B_2 \text{ and } \Lambda_1 \leq_d \Lambda_2, \tag{10}$$

$$\text{then} \quad \Phi_1 \leq_d \Phi_2 \tag{11}$$

$$\text{and} \quad \mathbb{E}(F_1) \leq \mathbb{E}(F_2) \tag{12}$$

$$\text{(ii)} \quad \text{If} \quad A_2 \leq_{icv} A_1, B_1 \leq_{icx} B_2 \text{ and } \Lambda_1 \leq_{icx} \Lambda_2 \tag{13}$$

$$\text{then} \quad \Phi_1 \leq_d \Phi_2 \text{ and } \mathbb{E}(F_1) \leq \mathbb{E}(F_2) \tag{14}$$

Proof.

(i) From section 2, it appears that the flow time may be expressed in terms of the classical measures and the lead-time

$$F = \max(S, L) = \max(w + \tau, L) = \Psi(w, \tau, L).$$

By appealing again the theorem 5.2.1 of Stoyan [20], we have

$$K_1 \leq_d K_2 \ \Rightarrow W_1 \leq_d W_2 .$$

Since " $\leq_d$ " has the convolution property (proposition 3 §3) then

$$W_1 \leq_d W_2 \text{ and } B_1 \leq_d B_2 \ \Rightarrow V_1 \leq_d V_2 .$$

The mapping Ψ is non decreasing and convex, and by using the mapping method 2.2.2 of Stoyan [20], we have that (11) follows.

(ii) The conditions $A_2 \leq_{icx} A_1$ and $B_1 \leq_{icx} B_2$ are sufficient for $K_1 \leq_{icx} K_2$, to holds. The implication : $\Lambda_1 \leq_{icx} \Lambda_2$ and $K_1 \leq_{icx} K_2 \Rightarrow \Phi_1 \leq_d \Phi_2$ follows now by applying again the mapping method [20].

Theorem 3 *Assume that* Λ *is* **HNBUE**. *If* $\Phi(x) = 1-e^{-\theta x}$, *for some* $\theta \geq 0$, $\theta \neq \int_0^\infty [1 - \Lambda(x)]dx$, *then the sojourn time is exponentially distributed, with the same distribution of the flow time* $\Phi(x)$.

Proof. The proof is a consequence of a result of Jun Cain [4] who showed that if a parallel system with **HNBUE** components has exponential life, it is essentially a series system with one component. More precisely,

Lemma 1 *[Jun Cai (1994)]. Let* F *be the life of a parallel system with* n **HNBUE** *components; and* F_i *be the life distribution of component* i, $i = 1, \ldots, n$. *If* F *is exponential, then there exists some* i $(1 \leq i \leq n)$ *such that* $\bar{F}(t) = \bar{F}_i(t)$, $t \geq 0$.

By definition of the flow time (section2), we have that Λ is **HNBUE**, and Φ is **HNBUE**, so that , by lemma 1 : either $\Phi = S$, either $\Phi = \Lambda$. But by hypothesis, $\Phi \neq \Lambda$, so that, $\Phi = S$, which implies that S is exponentially distributed with same parameter θ as Φ.

Remark. The class of queueing models for which S belong the class of exponential distributions is indeed not empty as we will seen in section 6.

6 Bounding due-date-based performance measures.

In this section, we give an illustration of the preceding result to derive bounds upon the due-date-based performance measures of our queueing model.

Theorem 4 *Let* Σ *be a* $GI/G/1$ *queue with parametric distributions* $A(x)$ *and* $H(x)$, *then*

(i) If $H(x) = 1 - e^{-\mu(1-r)x}$, *and the lead-time distribution function is* **NBUE (NWUE)**, *then for tardiness distribution function*

$$T \leq_{icx} (\geq_{icx}) T_0 \qquad \text{and} \qquad \mathbb{E}(T) \leq \rho/\mu(1-r),$$

where T_0 *is a random variable such that* $T_0 = 0$ *with probability* $1-\rho$ *and* T_0 *is exponentially distributed at rate* $\mu(1-r)$ *with probability* $\rho = \nu/[\nu + \mu(1-r)]$.

(ii) If $V(.)$ and $\Lambda(.)$ are both **IFR** *, then*

$$\Phi(x) \leq \Phi_e(x) \qquad \text{for } x < \min(1/\nu, m) \tag{15}$$

where $\nu^{-1} = \int_0^\infty [1 - \Lambda(x)]dx$ and $m = \int_0^\infty [1 - V(x)]dx$ and $\Phi_e(x) = (1 - e^{-\frac{x}{m}})(1 - e^{-\nu x})$.

In particular, if only Λ is **IFR**, *then (15) holds for $GI/M/1$ with $m = \mu(1-r)$.*

Proof.

(i) Indeed, for the $GI/M/1$ queue, we know a closed-form of sojourn distribution function [16]

$$V(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-\mu(1-r)x} & x \geq 0, \end{cases} \tag{16}$$

where $r = \sum_{k=0}^{\infty} b_k r^k$ and b_k is defined to be the probability of completing exactly k orders within a given (continuously busy) interarrival period [16]. When the orders form a homogeneous Poisson process, then $r = \lambda/\mu$. When arrivals are not Poisson, we must resort to numerical methods to determine the value of r.

Denote by Σ_1 our system Σ with $\Lambda_1(x) = \Lambda(x)$, $\nu_1 = \nu$. On the other hand, let Σ_2 be a system with the same parametric distribution functions A, H, and for which $\Lambda_2(x) = 1 - e^{-\nu x}$(*i.e.* $\nu_2 = \nu$).

Since Λ_1 is **NBUE** , then $\Lambda \leq_{icx} Exp(\nu)$. But, from the theory of stochastic ordering (proposition 2(iii) §3), the convex and concave orders are related as follows :

$$If \qquad \mathbb{E}(X) = \mathbb{E}(Y) \text{ , then } \qquad X \leq_{icv} Y \Leftrightarrow Y \leq_{icx} X,$$

In our case : $\Lambda \leq_{icx}$ Exp $(\nu) \Leftrightarrow$ Exp $(\nu) \leq_{icv} \Lambda \Leftrightarrow \Lambda_2 \leq_{icv} \Lambda_1$,

By theorem 1, it follows now that $\Gamma \leq_{icv} \Gamma_2$, where Γ_2 may be easily evaluated

$$\Gamma_2(x) = \int_0^\infty [1 - e^{-\mu(1-r)(x+y)}]\nu e^{-\nu y} dy = 1 - \frac{\nu}{\mu(1-r)+\nu} e^{-\mu(1-r)x},$$

or $\Gamma_2(x) = 1 - \rho + \rho e^{-\mu(1-r)x}$, $\rho = \nu/[\nu + \mu(1-r)]$.

For the convex ordering, we have from proposition 1 §3 that $T \leq_{icx} T_0 \Rightarrow \mathbb{E}(T) \leq \mathbb{E}(T_0)$ which yields the second inequality.

(ii) Since the lead-time distribution in Σ is **IFR** , then by proposition 4 §3, we have that $\bar{\Lambda}(x) \geq e^{-\nu t}$ for $t < 1/\nu$ and $\bar{V}(x) \geq e^{-t/m}$ for $t < m$. Hence, the result follows now by (11) of theorem 2 §5.

Theorem 5 *Let Σ be an $GI/M/1$ queue. If V and Λ are* **IFRA (DFRA)** *then*

$$\mathbb{E}(F) \leq \frac{1}{\nu} + m - \frac{1}{\nu + 1/m}. \tag{17}$$

In particular, if Σ is an $GI/M/1$ queue, and only Λ is IFRA, then (17) holds with $m = 1/\nu(1-r)$.

The proof is a consequense of (14) by using the properties of IFRA distribution functions (see also Barlow and Proshan [1] corollary 7.7 p.123) :

$$\mathbb{E}(F) \leq (\geq) \int_0^{\infty} [1 - (1 - e^{-t/m})(1 - e^{-\nu t})]dt = \frac{1}{\nu} + m - \frac{1}{\nu + 1/m}.$$

7 Discussions and applications.

The results of section 6 illustrates the comparability method developed in section 5 for Queuing models which incorparates lead-time as parameter. They are useful in understanding a complex system with unknown distribution of performance measures as tardiness or flow time. This approach leads to bounds on these performance measures based on partial information about parametric distributions.

Usually, by way of partial information about parametric distribution, it is assumed the knowledge about a certain number of moments (often, the first two moments) [Kimura (1987), White (1986)]. Here, we take another look by assuming that these distributions belong to some nonparametric class of distributions. The interest of these distributions is to show a certain deviation from the exponentiality characterised by some ageing property.

7.1 Relation between exponential and ageing distributions.

From the classification diagram 1 (appendix B), we see that the exponential distribution possess all the ageing properties. However, each class is strictly larger than each of preceding one in the diagram.

Example 1. The NBUE class is related to, but contains and is much larger than the NBU class. A counter example is given in Klefsjo. The distribution which put mass of 0, 5 at the point $x = 1$, and mass 0, 5 , at $x = 3$, is NBUE, but not NBU.

Example 2. The distribution $F(t) = 1 - e^{[t]}$ is perhaps not frequent in the nature. Barlow &Proshan (1975) gave an example in which such a distribution occurs (handling an airplane subject to shocks). This Distribution is NBU from the inequality $[a+x] \geq [a] + [x]$. But, it is not IFRA since $[t]/t$ is not increasing.

Example 3. The distribution $F(t) = (1 - e^{-t})(1 - e^{-2t})$ describes the reliability of a system with two exponential components 1 and 2 respectively. It can be shown that F is IFRA, but the failure rate is increasing until some fixed value, and then decreases. Thus F is not IFR.

Example 4. Any distribution with strictly increasing failure rate is IFR but nonexponential. For example, the Raighley distribution $F(x) = 1 - exp(-cx^2)$ with linear rate and non null slope.

From these examples, we see that we can find non exponential distribution in each class of ageing distributions through some fixed ordering. Let F be a distribution function and E denotes the exponential distribution, then

F is IFR if and only if $F \leq_{icx} E$, where $\leq_{icx}$ is the increasing convex ordering.

F is IFRA if and only if $F \leq_{*} E$, where $\leq_{*}$ is the star-shaped ordering.

F is NBU if and only if $F \leq_{su} E$, where $\leq_{su}$ is the super-additive ordering.

7.2 Some applications of theorem 4.

Now the results of theorem 4(i) show that if we have the information that the service time is exponential and the lead-time is NBUE, then the tardiness of our unknown system is less (in the increasing convex ordering sense) than an exponential random variable at rate $\mu(1-r)$, where r is defined in section 4. Note that this bound is also valid if the parametric distributions A and B are such that the sojourn time is NBUE, since in this case this distribution is bounded (in the increasing convex order sense) by an exponential distribution. This bound may also be used for arbitrary queues in heavy traffic for which we can obtain exponential approximation similar to that of Kingman; see for example Kleinrock (1976). In this case we must replace the parameter $\mu(1-r)$ by the parameter corresponding to the diffusion approximation.

If the parametric distributions A and B are arbitrary, but such that V is IFR, and if we have additional information that the lead-time is also IFR, then the theorem 4(ii) shows that the flow time distribution is less than $\Phi_e(x) = (1 - e^{-x/m})[1 - e^{-\nu x}]$ for each value $x \leq \min(1/\nu, m)$. Such a bound is interesting since it depends only on the first moments m and $1/\nu$, provided the mean sojourn time is known.

	$\mu \downarrow r \rightarrow$	0.1	0.2	0.5	0.7	0.8	0.9
	1,1						
Upper bound		0,3389	0,4117	0,8658	1,8254	3,1565	7,4515
Exact value		0,2135	0,2762	0,6990	1,6522	2,9985	7,3398
	1,2						
Upper bound		0,2930	0,3567	0,7575	1,1649	2,8153	6,7200
Exact value		0,1753	0,2286	0,5950	1,4406	2,6528	6,6024
	1,5						
Upper bound		0,2002	0,2450	0,5333	1,1695	2,0833	5,1282
Exact value		0,1024	0,1368	0,3851	0,9965	1,9123	4,9944
	1,8						
Upper bound		0,1455	0,1789	0,3968	0.8903	1.6149	4.0849
Exact value		0,0635	0,0868	0,2632	0,7228	1,4406	3,9389
	2						
Upper bound		0,1207	0,1488	0,3333	0,7575	1,3888	3,5714
Exact value		0,0471	0,0656	0,2088	0,5950	1,2143	3,4188
	5						
Upper bound		0,0222	0,0277	0,0666	0,1666	0,3333	1,0000
Exact value		0,0012	0,0022	0,0165	0,0780	0,2088	0,8296
	10						
Upper bound		$5,8.10^{-3}$	$7,3.10^{-3}$	$1,8.10^{-2}$	$4,7.10^{-2}$	0,1	0,3333
Exact value		$6,8.10^{-6}$	$2,1.10^{-5}$	$6,7.10^{-4}$	$8,3.10^{-3}$	$3,4.10^{-2}$	0,2088
	30						
Upper bound		$6,7.10^{-4}$	$8,5.10^{-4}$	$2,1.10^{-3}$	$5,8.10^{-3}$	$1,2.10^{-2}$	$4,7.10^{-2}$
Exact value		$3,4.10^{-14}$	$7,8.10^{-13}$	$.10^{-8}$	$6,8.10^{-6}$	$2,0.10^{-4}$	$8,3.10^{-3}$
	50						
Upper bound		$2,4.10^{-4}$	3.10^{-4}	$7,8.10^{-4}$	$2,1.10^{-4}$	$7,4.10^{-4}$	$1,8.10^{-2}$
Exact value		$3,1.10^{-22}$	$5,3.10^{-20}$	$2,7.10^{-13}$	10^{-8}	$2,2.10^{-6}$	$6,7.10^{-4}$
	100						
Upper bound		$6,1.10^{-5}$	$7,7.10^{-5}$	$1,9.10^{-4}$	$5,4.10^{-4}$	$1,2.10^{-3}$	$4,7.10^{-3}$
Exact value		$4,5.10^{-42}$	$1,1.10^{-37}$	$1,9.10^{-24}$	$1,5.10^{-15}$	$5,1.10^{-11}$	$2,2.10^{-6}$

Table 1: Upper bound on the mean Tardiness time when the service time is exponential and the lead-time is NBUE (Example 1) (system $G/M/1$)

Similarly, if the service time is exponential with arbitrary interarrival distribution such that V is IFRA, and if we have the additional information that the lead-time is IFRA, then the mean flow time is bounded by $(\nu^{-1} + m - (\nu + m^{-1}))^{-1}$, which bound can be easily computed since again it depends only on the first moments.

To illustrate the effect of nonexponential service, we have also computed a bound on mean tardiness time for a $G/E_k/1$ queue for which the distribution of the sojourn time can be evaluated at least numerically, and sometimes explicitely. Calculation of the distribution of the sojourn time and the bound on the mean tardiness time are reported in appendix C. For lack of space we have restricted ourselves to the case of Erlangian distribution of order 2 since in this case, the parameter α_i and z_i can be evaluated explicitly. For $k \geq 2$ we must resort to numerical methods.

	$\mu \downarrow r \rightarrow$	0.1	0.2	0.5	0.7	0.8	0.9
	1,1						
Upper bound Exact value		0,4507 0,1421	0,5403 0,3691	1,0707 0,7160	2,1436 1,8330	3,5630 3,3251	7,9804 6,7707
	1,2						
Upper bound Exact value		0,3934 0,1046	0,4728 0,1429	0,9513 0,5883	1,9141 1,5868	3,2031 2,9561	7,2436 7,0935
	1,5						
Upper bound Exact value		0,2751 0,0317	0,3328 0,0551	0,6872 0,3305	1,4208 1,0754	2,4224 2,1308	5,6116 4,4267
	2						
Upper bound Exact value		0,1706 $1,1.10^{-3}$	0,2079 $7,2.10^{-3}$	0,4437 0,1255	0,8311 0,5883	1,6651 1,3329	3,9997 3,7703
	5						
Upper bound Exact value		0,0334 $4,1.10^{-6}$	0,0415 $2,3.10^{-5}$	0,0967 $8,3.10^{-3}$	0,2314 0,0131	0,4407 0,1255	1,2295 0,8751
	10						
Upper bound Exact value		9.10^{-3} $4,1.10^{-6}$	$1,3.10^{-2}$	$2,7.10^{-2}$	7.10^{-2} $2,9.10^{-5}$	0,1425	0,4437 0,1255
	50						
Upper bound Exact value		$3,8.10^{-4}$	....	....	....	....	....

Table 2: Upper bound on the mean Tardiness time when the service time is exponential and the lead-time is IFR with Raighley distribution (Example 4) (system $G/M/1$)

8 Numerical Examples.

In this section, we give several illustrations of the preceding results. In our first numerical examples, we have considered a $G/M/1$ queue with exponential service time and NBUE lead-time distribution. In this case, the tardiness is bounded by an exponential random variable at rate $\mu(1-r)$. This bound corresponds to a simpler system with exponential lead-time. Table 1 gives selected numerical values of the upper bound on mean tardiness time for different values of r and $\rho = \lambda/\mu$ (we have fixed $\lambda = 1$ and let varying the service rate μ).

	$\mu \downarrow r \rightarrow$	0.1	0.2	0.5	0.7	0.8	0.9
	0,1						
Upper bound Exact value		11,2219 11,0329	12,5995 12,5846	20,064 20,054	33,3727 33,3665	50,0266 50,0223	100,0134 100,0112
	0,5						
Upper bound Exact value		2,6238 2,5809	2,8712 2,8218	4,2634 4,2299	6,8404 6,8161	10,1218 10,1039	20,0643 20,0543
	1						
Upper bound Exact value		1,7086 1,6630	1,8132 1,7671	2,4298 2,3857	3,6358 3,5990	5,2207 5,1912	10,1218 10,1039
	2						
Upper bound Exact value		1,3458 1,3102	1,3846 1,3466	1,6282 1,5833	2,1470 2,1014	2,8712 2,8298	5,2207 5,1912
	5						
Upper bound Exact value		1,2022 1,0198	1,2107 1,1865	1,2687 1,2405	1,4090 1,3698	1,6282 1,5833	2,4298 2,3857
	10						
Upper bound Exact value		1.1763 1,1702	1,1787 1,1746	1,1959 1,1821	1,2407 1,2166	1,3166 1,2833	1,6282 1,5833
	30						
Upper bound Exact value		1,1678 1,1668	1,1681 1,1669	1,1702 1,1675	1,1763 1,1702	1,1875 1,1765	1,2407 1,2166
	50						
Upper bound Exact value		1,1670 1,1667	1,1672 1,1667	1,1679 1,1669	1,1702 1,1675	1,1745 1,1693	1,1959 1,1821
	100						
Upper bound Exact value		1,1667 1,1666	1,1668 1,1666	1,1670 1,1666	1,1675 1,1667	1,1687 1,1670	1,1745 1,1693

Table 3: Upper bound on the mean Flow time when the service time is exponential and the lead-time is IFRA (system $G/M/1$)

For the sake of comparison, we also gives the exact values for a system with a strictly NBUE distribution which is not NBU (and not exponential) (see example 1 of section 7). The upper bound for $\mathbb{E}(T)$ increases for fixed value of μ or ρ. Note that in this case, the mean sojourn time also increases. The exact value of the mean tardiness time is more closed to the upper bound for great values of μ and r. We have also provided computations for the distribution of example 4 which is strictly IFR, thus NBUE, but nonexponential (table 2). The same conclusion have been noted.

In the second numerical example (table 3), we studied the variation of the upper bound on the mean flow time when the sojourn time and the lead-time are both IFRA. The numerical comparisons was made with exact values for a system with the IFRA distribution of example 3 which is not IFR, nor

	$\mu \downarrow r \rightarrow$	0.01	0.1	0.5	1	10	100
	0,1						
Upper bound Exact value		$2,1.10^{-4}$ $1,1.10^{-6}$	2.10^{-3} 10^{-6}	$9,89.10^{-3}$ $2,21.10^{-3}$	$1,2.10^{-3}$ $1,1.10^{-3}$	$8,8.10^{-2}$ $8,3.10^{-2}$	0,1342 0,1342
	0,5						
Upper bound Exact value		$1,5.10^{-2}$ $6,1.10^{-4}$	0,1397 0,0442	0,5067 0,3903	0,7500 0,6800	1,2922 1,2914	1,3804 1,3803
	0,8						
Upper bound Exact value		10^{-2} 4.10^{-2}	0,0921 0,0296	3,00 2,88	3,5973 3,5577	4,3608 4,3606	4,4503 4,4503
	0,9						
Upper bound Exact value		0,8000 0,2256	4,5899 4,0060	7,9321 7,8462	8,7070 8,6858	9,5463 9,5462	9,6362 9,6362

Table 4: Variation of the upper bound on the mean Tardiness time when the service time is Erlangian of order 2 and the lead-time is NBUE (appendix) (system $G/E_k/1$)

exponential. We note that the upper bound increases when r increases for fixed μ, and rapidly decreases when μ increases for fixed r. For great value of μ, the upper bound is very closed to the exact values.

Next, we considered a system with arbitrary interarrival time distribution and k-Elangian service distribution for which the sojourn time distribution is known (see appendix C). In this case the upper bound is obtained from formulas (C). In the numerical example, we have shown comparisons with exact values when the lead-time is 2-Erlang (this distribution is IFR thus NBUE). In table 4, we study the variation of ν and ρ. The bound increases when ν increases and tends to be more closed to the exact values for great values of ν.

Finally, in figures 1-4 we have plotted a family of curves which represents the bound $\Phi_e(x)$ on flow time distribution as a function of x for different values of the mean lead-time $1/\nu$, in the case of a system with IFR lead-time distribution. In figures 1-3, we have fixed the mean lead-time $\nu = 0,25$ and we compare the upper bound $\Phi_e(x)$ with the exact value in the case of an $E_2/E_2/1$ queue and IFR lead-time distribution for different values of $\rho = 0,25$, $\rho = 0,5$ *and* $\rho = 0,8$. We note that the exact value tends to be more closed to the upper bound for great value of ρ. Figure 4 shows the variation of the upper bound $\Phi_e(x)$ on the flow time distribution when varying the mean lead-time.

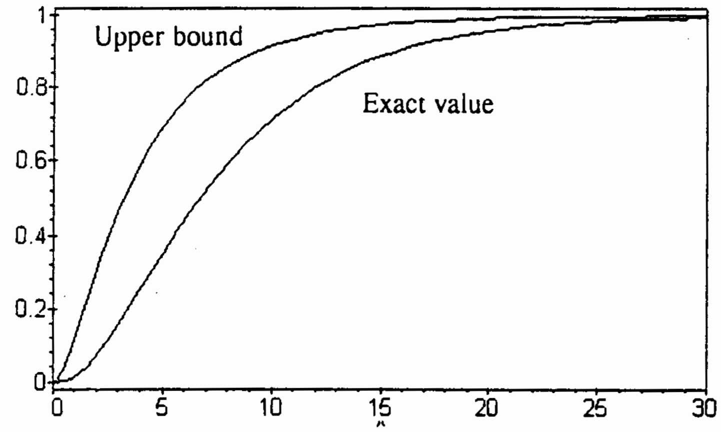

Figure 1. Upper bound $\phi_c(x)$ on the Flowtime distribution function when the mean leadtime $1/\nu=4$ for an intensity $\rho=0.5$.

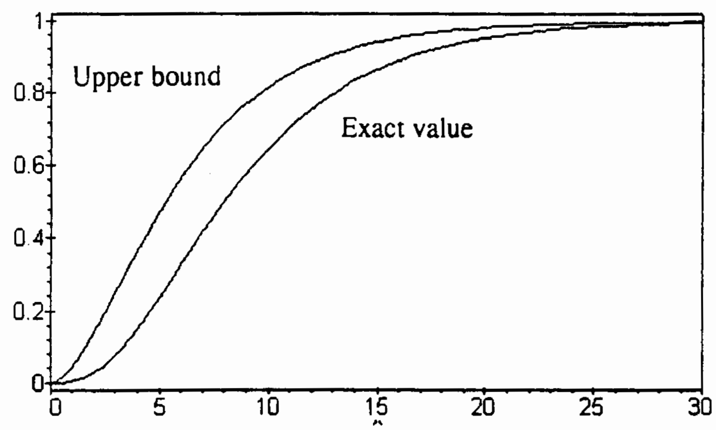

Figure 2. Upper bound $\phi_c(x)$ on the Flowtime distribution function when the mean leadtime $1/\nu=4$ for an intensity $\rho=0.8$.

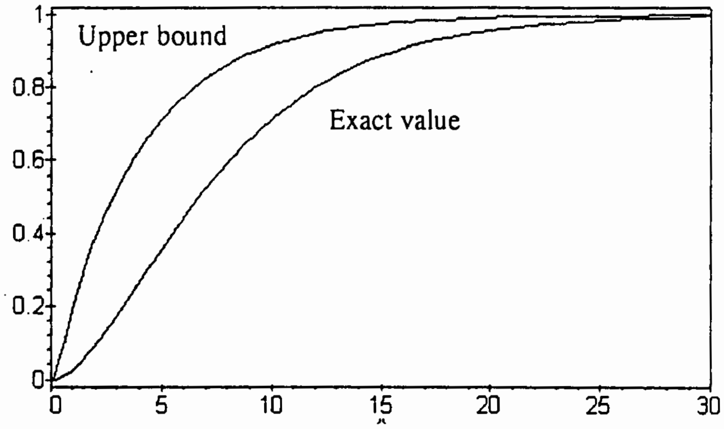

Figure 3. Upper bound $\phi_e(x)$ on the flowtime distribution function when the mean leadtime $1/\nu=4$ for an intensity $\rho=0.25$.

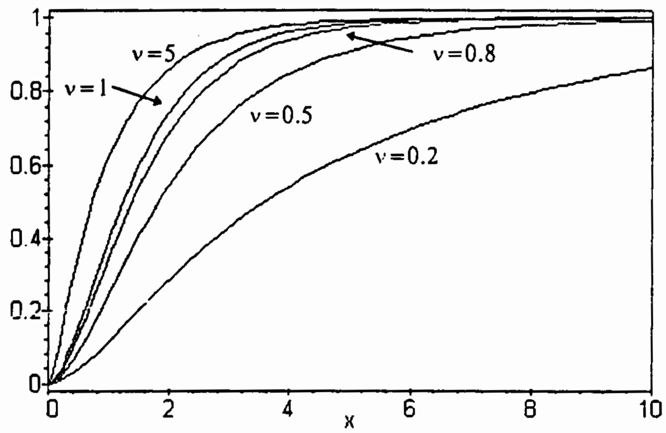

Figure 4. Variation of the upper bound $\phi_e(x)$ on the Flowtime distribution function versus the mean leadtime ν.

9 Conclusion.

This work has been motivated by the fact that many practical studies tried to incorporate due-date in their modelling approaches. The aim of this paper were to show how we can study the influence of the lead-time and other parameters upon due-date performance measures by using the theory of ageing distributions and comparability theory. For this purpose, and since this work don't intend to an exhaustive investigation of all related aspects, we have restricted ourselves to the framework of [21] which is quite simple and straightforward.

Our main results are provided in section 5 where we give conditions under parametric distribution (interarrival time, service time, lead-time) for which two such queueing models are comparable from the point of view of some due date performance measures as tardiness or flow time. This yields bounds on these performance measures which are useful in understanding unknown models by more simpler models for which an evaluation can be made. The accuracy of the results are provided in selected numerical examples.

Although we have opted for a simple way of modelling due-date, this study shows that the comparability theory and ageing distributions intensively used in the last decades in Reliability theory, gives an interesting approach in understanding such models. It seems that this approach can be extended to other performance measures as Work-in-Processes (WIP) and / or queue disciplines (priority, breakdowns, retrials, vacations, ...). We hope that the results of section 5 holds for more strong stochastic orders described in the literature. For example, it is not difficult to prove that it holds for the convex orders since the main used properties (lemma 1 and 3) are valid for this order. It seems that it also holds for the likelihood orders, but the proof seems to be more difficult.

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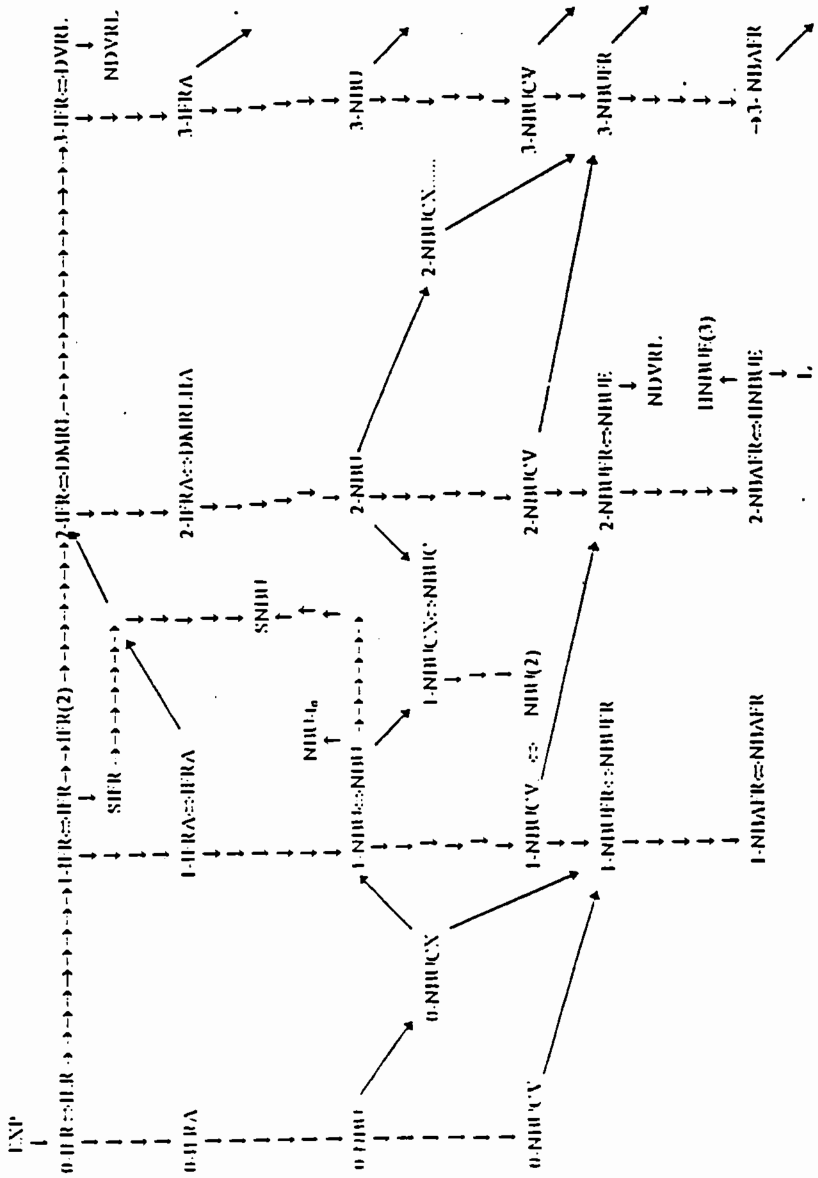
Appendix A. Acronyms for ageing distributions.

IFR	Increasing Failure Rate
SIFR	Stochastically IFR
IFRA	Increasing Failure Rate Average
DMRL	Decreasing Mean Residual Life
NBU	New Better than Used
NBUCX	New Better than Used in Convex Ordering
NBUCV	New Better than Used in Concave Ordering
SNBU	Stochastically New Better than Used
NBU-t_0	New Better than Used of age t_0
NBUE	New Better than Used in Expectation
HNBUE	Harmonically New Better than Used in Expectation
NBUFR	New Better than Used in Failure Rate
NBUFRA	New Better than Used in Failure Rate Average
NBAFR	New Better than Average Failure Rate (the same as NBUFRA)
BMRL-t_0	Better Mean Residual Life at t_0.
DVRL	Decreasing Variance of Residual Life
NDVRL	Net DVRL
DPRL-α	Decreasing 100% percentile Residual Life
NBUP-α	New Better than Used with Respect to the 100% percentile
ILR	Increasing in Likelihood Rate
IHR	Increasing in Hazard Rate (the same as IFR)
DMRLHA	Decreasing Mean Residual Life in Hazard Average

Appendix B. Classification of ageing distributions.



Appendix C. Sojourn time distribution in $G/E_k/1$ queue.

From the general queueing theory, it is well known that for the $G/E_k/1$ queue, the waiting time distribution is given by the formula

$$W(x) = 1 - \sum_{i=1}^{k} A_i e^{-\mu(1-z_i)x}$$

where

$$A_i = \frac{\alpha_i z_i}{1-z_i} / \sum_{i=1}^{k} \frac{\alpha_i}{1-z_i}$$

$$\alpha_i = \prod_{i=1_{j\neq i}}^{k} \frac{z_i}{z_i - z_j}$$

The quantity $z_1, \ldots, z_k$ are solutions in the disk $|z| \leq 1$ of the equation

$$z^k = F(z)$$

where

$$F(z) = \int_0^\infty e^{-\mu y(1-z)} dA(y)$$

For the sake of illustrations we considered the case of 2-Erlang distributions

$$B(x) = 1 - (1 + \mu x)e^{-\mu x}$$

$$A(x) = 1 - (1 + \lambda x)e^{-\lambda x}$$

$$\Lambda(x) = 1 - (1 + \nu x)e^{-\nu x}$$

The solutions of the equation (C.1) are $1, \rho, \frac{1+\rho\pm\sqrt{\Delta}}{2}$ where $\Delta = 1+\rho^2+6\rho > 0$. We retain only the solutions $z_1 = \rho$ and $z_2 = \alpha = \frac{1+\rho-\sqrt{\Delta}}{2}$ which are in the disk $|z| \leq 1$.

Thus

$$W(y) = 1 - \frac{\rho^2(1-\alpha)}{\rho - \alpha} e^{-\mu(1-\rho)y} - \frac{\alpha^2(1-\rho)}{\alpha - \rho} e^{-\mu(1-\alpha)y}$$

We thus obtain the sojourn time distribution

$$V(y) = 1 - \frac{1-\alpha}{\rho - \alpha} e^{-\mu(1-\rho)y} - \frac{1-\rho}{\alpha - \rho} e^{-\mu(1-\alpha)y}$$

From formula (9), we can found the tardiness distribution

$$\Gamma(x) = 1 - \nu^2 \frac{1-\alpha}{\rho-\alpha} \frac{e^{-\mu(1-\rho)x}}{[\nu + \mu(1-\rho)]^2} - \nu^2 \frac{1-\rho}{\alpha-\rho} \frac{e^{-\mu(1-\alpha)x}}{[\nu + \mu(1-\alpha)]^2}$$

and the exact value of mean tardiness time is thus

$$\gamma = \frac{\nu^2}{\mu(\rho-\alpha)} [\frac{1-\alpha}{(1-\rho)[\nu + \mu(1-\rho)]^2} - \frac{1-\rho}{(1-\alpha)[\nu + \mu(1-\alpha)]^2}]$$