An Algorithm for Mixed Integer Linear Fractional Programming Problems

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Abstract: *A new algorithm for solving mixed integer linear fractional programming problem which maximizes a linear fractional objective function under the constraint of some linear inequalities is developed in the present paper . After obtaining an optimal solution of the continuous fractional problem whatever the feasible region is, we are able to construct an improved branch and bound method based on computation of penalties for solving the mixed integer linear fractional problem in a finite number of iterations. An illustrative numerical example is included.*

Key Words: *linear fractional programming, non linear programming, integer programming.*

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1 Introduction

Linear fractional programming problems have applications in various field such as the cutting stock problem [7], the travelling salesman problem, Markovian replacement problems [10] and portfolio theory [21]. For more applications of fractional programming see Stancu-Minasian's book [17].

It has been treated by several authors (e.g. Abbas and Moulaï [1], Bazara and Shetty [2], Cambini and Martein [4], Charnes and Cooper [6], Granot [8], Martos [11], Seshan and Tikekar [14], Stancu-Minasian ([15],[16]) and Williams [20]).

Charnes-Cooper [6] and Martos [11] developed methods for solving the continuous problem. Cambini and Martein [4] modified the Martos' approach [11] to deal with unbounded feasibility region.

Chandra and Chandramohan [5] developed a method based on Charnes and Cooper transformation to solve the mixed integer linear fractional programming problem using a branch and bound technique.

Seshan and Tikekar [14] suggested two algorithms based on a parametric approach and Bitran-Novaes [3] idea to solve integer linear fractional programming problems. Abbas and Moulaï [1], however, solve the same problem by branch and bound technique.

The method that we are going to present maintain the original format of the problem without any transformation and can be considered as an alternative method to Granot [8] and Seshan and Tikekar [14] algorithms for any feasible region.

The present paper is organized as follows : in Section 2, we introduce the necessary definitions and notation. Section 3 describes our methodology that can be used to solve mixed integer linear fractional programming problems. A numerical example is presented to illustrate the proposed method.

2 Notations

Let S the set of vectors $x \in R^n$ satisfying the constraints $x \geq 0$ and $Ax \leq b$ where A is a real $m \times n$ matrix and b a vector of R^m. Let p, q vectors of R^n and α, β two elements of R and L is a subset of $\{1, 2, ..., n\}$.

The mixed integer linear fractional programming problem $(\mathcal{P})$ intended to be studied can be mathematically stated as:

$$\begin{aligned} &\text{Maximize } Z(x) = \frac{p'x + \alpha}{q'x + \beta} \qquad (1) \\ &\text{Subject to} \\ &\qquad x \in S \\ &\qquad x_j \text{ integers, } j \in L \end{aligned}$$

we assume that the denominator $q'x+\beta$ is non negative over S. The relaxed problem $(\mathcal{P}_1)$ is the problem $(\mathcal{P})$ without integer constraints.

In the following section, we present the method in detail with a convergence theorem, and a numerical illustration.

3 The Mixed Integer Linear Fractional Programs

In this section, a branch and bound algorithm for solving mixed integer linear fractional programs $(\mathcal{P})$ is presented. The algorithm is based on Cambini and Martein's method for solving continuous fractional programs. The original format of the objective fractional function and original structure of the constraints are maintained and the iterations are carried out in an augmented simplex tableau which includes m+3 rows. The first m rows correspond to the original constraints, the m+1 and m+2 rows corresponds to the numerator and the denominator of the objective fractional function, respectively, and the last row correspond to the $\bar{\gamma}_j$'s where

$$\bar{\gamma}_j = \overline{\beta} \times \overline{p}_j \; - \overline{\alpha} \times \overline{q}_j, \qquad j \in I_N \tag{2}$$

where I_N is the index set of the non-basic variables.

In every iteration of the algorithm, the first m+2 rows are modified as usual through the pivot operations, whereas the last row is modified via the $\bar{\gamma}_j$'s formula.

The non-basic variable which must enter to the basis is indicated by the index k such that

$$\frac{\bar{p}_k}{\bar{q}_k} = \max\left\{\frac{\overline{p}_j}{\overline{q}_j} : j \in I_N; \bar{q}_j > 0\right\} \tag{3}$$

The solution of the continuous fractional problem obtained in a finite sequence of basic optimal level solutions (e.g., [4]) is optimal if and only if $\bar{\gamma}_j \leq 0$ for all $j \in I_N$. Otherwise, there exists an index k for which $\bar{\gamma}_j > 0$. If the components of the column vector associated to index k are negative, then the maximum of problem $(\mathcal{P}_1)$ is not attained but its supremum is given by:

$$\sup_{x \in S} Z(x) = \frac{\bar{p}_k}{\bar{q}_k} \tag{4}$$

This is briefly the Cambini and Martein's method for solving continuous fractional problem which is used at the beginning of the proposed branch and bound procedure.

3.1 Methodology for solving MILFP

By using the Charnes and Cooper transformation, several authors (e.g. [5], [8], [14]) develop methods for solving the problem ($\mathcal{P}$) assuming always that the feasible region is bounded. If is not the case, they maintain the original format for the objective function and constraints but they use others optimization methods as cutting plane techniques [8] for a bounded feasible region too. The difference between their procedures and the algorithm that we are going to describe is the fact that we solve ($\mathcal{P}$) by maintaining the original format of the problem whatever the feasible region is, using computation of penalties which improve considerably the branch and bound technique.

Consider the problem ($\mathcal{P}_1$) by introducing slack variables and solving it by the Cambini and Martein's method to find an optimal continuous solution. Let the optimal simplex tableau be given by ($\mathcal{P}_1'$):

$$\text{Maximize } Z(x) = \frac{\bar{\alpha} + \sum_{j\in I_N} \bar{p}_j\, x_j}{\bar{\beta} + \sum_{j\in I_N} \bar{q}_j\, x_j} \tag{5}$$

Subject to

$$x_i + \sum_{j\in I_N} \bar{a}_{ij}\, x_j = e_i, \quad i \in I_B \tag{6}$$

$$x_j \geq 0 \ , \quad j \in I_N$$

where I_N is the index set of the non-basic variables, x_i, $i \in I_B$ is the basic variable, $\bar{\alpha}$ and $\bar{\beta}$ are the reduced costs in the simplex tableau and $\bar{\alpha}/\bar{\beta}$ is the value of the objective function. The optimal basic solution of problem ($\mathcal{P}_1$) is given by:

$$x_i = e_i,\ i \in I_B$$

$$\text{otherwise} \quad x_j = 0, \quad j \in I_N$$

If e_i is integer for every $i \in L \cap I_B$, then the required solution to problem ($\mathcal{P}$) is obtained.

If the necessary integrality restrictions are not satisfied, let e_{kt} be non integer value of x_{kt} for some $kt \in L \cap I_B$. We denote the largest integer less than e_{kt} by $\lfloor e_{kt} \rfloor$ and the smallest integer greater than e_{kt} by $\lceil e_{kt} \rceil$. Since x_{kt} is required to be integer, either $x_{kt} \leq \lfloor e_{kt} \rfloor$ or $x_{kt} \geq \lceil e_{kt} \rceil$.

Let us consider the former $x_{kt} \leq \lfloor e_{kt} \rfloor$ which gives rise to the constraint $x_{kt} + s = \lfloor e_{kt} \rfloor$ but $x_{kt} + \sum_{j\in I_N} \bar{a}_{kt,j}\, x_j = e_{kt}$ from the simplex tableau. Then

$$- \sum_{j\in I_N} \bar{a}_{kt,j}\, x_j + s = \lfloor e_{kt} \rfloor - e_{kt}.$$

Thus we have

$$- \sum_{j \in I_N} \bar{a}_{kt,j} \, x_j \leq \lfloor e_{kt} \rfloor - e_{kt} \tag{7}$$

It is obvious that $\lfloor e_{kt} \rfloor - e_{kt}$ is negative and the optimal solution to problem $(\mathcal{P})$ given below does not satisfy the constraint (7). Augmenting this constraint to problem $(\mathcal{P})$, we obtain one of the branches.

Similarly, corresponding to $x_{kt} \geq \lceil e_{kt} \rceil$, we obtain the constraint :

$$-x_{kt} + s = -\lceil e_{kt} \rceil.$$

Then

$$\sum_{j \in I_N} \bar{a}_{kt,j} \, x_j \leq e_{kt} - \lceil e_{kt} \rceil < 0. \tag{8}$$

Introducing this constraint to problem $(\mathcal{P})$, we obtain the other branch. This, together with the rules for the selection of branching variables completely describes the branching strategy of the method.

3.2 Computation of Penalties

After having obtained the optimal continuous solution of $(\mathcal{P}_1)$, the associated objective fractional function in the optimal simplex tableau is given by equation (5).

Let e_{kt} be a non integer value of x_{kt} for some $kt \in L \cap I_B$. For selecting a new branch which must be added to problem $(\mathcal{P}'_1)$, we compute the penalties π_r and $\pi_{r'}$ of the constraints $x_{kt} \leq \lfloor e_{kt} \rfloor$ and $x_{kt} \geq \lceil e_{kt} \rceil$, respectively, given by:

$$\pi_r = \frac{e.\Delta_r}{\bar{\beta} \, (\bar{\beta} + \dfrac{e. \, \bar{q}_r}{\bar{a}_{kt,r}})} \tag{9}$$

and

$$\pi_{r'} = \frac{(1-e).\Delta_{r'}}{\bar{\beta} \, (\bar{\beta} - \dfrac{(1-e). \, \bar{q}_{r'}}{\bar{a}_{kt,r'}})} \tag{10}$$

where

$$\Delta_r = \min \left\{ \frac{\bar{\gamma}_j}{- \, \bar{a}_{kt,j}} / \, \bar{a}_{kt,j} > 0 \right\} \tag{11}$$

$$\Delta_{r'} = \min \left\{ \frac{\bar{\gamma}_j}{\bar{a}_{kt,j}} / \, \bar{a}_{kt,j} < 0 \right\} \tag{12}$$

and

$$e = e_{kt} - \lfloor e_{kt} \rfloor. \tag{13}$$

The branch corresponding to the smallest penalty is augmented to problem $(\mathcal{P}'_1)$ which is solved by the following algorithm. This is much important for skipping many non promising branches and gaining considerable computing time.

3.3 The Algorithm

Step 1. Find an optimal continuous solution x^0 of problem $(\mathcal{P}_1)$

1. If a such solution does not exist, stop. Either

$$\sup_{x\in S} Z(x) = +\infty$$

   when the basic optimal level solution of $(\mathcal{P}_1)$ does not exist or

$$\sup_{x\in S} Z(x) = \max_{\bar{q}_j>0} \frac{\bar{p}_j}{\bar{q}_j}$$

   when a pivot operation is not possible.

2. Otherwise, set $k = 1$, $l = 0$ and go to step 2.

Step 2.

1. If x_j^0 is integer for all $j \in L \cap I_B$, stop. x^0 is an optimal solution of $(\mathcal{P})$.
2. Otherwise, let x_{kt} be, for some $kt \in L \cap I_B$, a non integer component of x^0 with corresponding value e_{kt}. Set $\pi_l = 0$ and go to step 3.

Step 3.

* Compute π_{2k-1} and π_{2k}. Let $\pi_{2k-1} = \pi_{2k-1} + \pi_l$, $\pi_{2k} = \pi_{2k} + \pi_l$ and $\pi_l = +\infty$.
* Compute

$$\pi_l = \min_{1\leq j\leq 2k} \{\pi_j\}\,.$$

* Augment the constraint to the optimal simplex tableau, solve it and go to step 4.

Step 4.

Let x^l an optimal solution of the augmented problem.

1. If x_j^l is integer for all $j \in L\cap I_B$, stop. x^l is the optimal solution of problem $(\mathcal{P})$.
2. Otherwise,
   (a) The augmented problem has no solutions, stop.
   (b) Let x_{kt}^l be a non integer component of x^l, $kt \in L\cap I_B$ with corresponding value e_{kt}^l. Set $k = k + 1$ and go to step 3.

Theorem 1 *The branch selected with the smallest penalty given by*

$$\pi_l = \min_{1\leq j\leq 2k} \{\pi_j\}$$

corresponds to the optimal value of the objective function for the all pendant vertices j of the rooted tree.

Proof
At the k^{th} iteration, we are on the vertex s_k of the rooted tree with

$$\widehat{Z}_l = \max_{1\leq j\leq 2k-2}\left\{\widehat{Z}_j\right\} = \widehat{Z}_{opt} - \min_{1\leq j\leq 2k-2}\{\pi_j\},$$

where $\widehat{Z}_{opt}$ is the optimal value of the objective function for the relaxed problem $(\mathcal{P}_1)$ and

$$\pi_l = \min_{1\leq j\leq 2k-2}\{\pi_j\}$$

is the smallest decrease of $\widehat{Z}_{opt}$ on all pendant vertices j of the rooted tree.

Assume that the solution x_k is not integer, we must choose a branch for solving the augmented problem. For this, we compute the penalties π'_{2k} and π'_{2k-1} on the vertex s_k for selecting the branch and we obtain $\widehat{Z}_{2k} = \widehat{Z}_l - \pi'_{2k}$ and $\widehat{Z}_{2k-1} = \widehat{Z}_l - \pi'_{2k-1}$. Thus $\widehat{Z}_{2k} = \widehat{Z}_{opt} - \pi_l - \pi'_{2k}$ and $\widehat{Z}_{2k-1} = \widehat{Z}_{opt} - \pi_l - \pi'_{2k-1}$ and we obtain : $\widehat{Z}_{2k} = \widehat{Z}_{opt} - \pi_{2k}$ and $\widehat{Z}_{2k-1} = \widehat{Z}_{opt} - \pi_{2k-1}$. Now, the problem is to choose the greatest value of the objective function on all pendant vertices of the rooted tree :

$$\begin{aligned}
\max_{1\leq j\leq 2k}\left\{\widehat{Z}_j\right\} &= \max\left\{\max_{1\leq j\leq 2k-2}\left\{\widehat{Z}_j\right\}, \widehat{Z}_{2k-1}, \widehat{Z}_{2k}\right\}\\
&= \max\left\{\widehat{Z}_l, \widehat{Z}_{2k-1}, \widehat{Z}_{2k}\right\}\\
&= \max\left\{\widehat{Z}_{opt} - \pi_l, \widehat{Z}_{opt} - \pi_{2k-1}, \widehat{Z}_{opt} - \pi_{2k}\right\}\\
&= \widehat{Z}_{opt} + \max\{-\pi_l, -\pi_{2k-1}, -\pi_{2k}\}\\
&= \widehat{Z}_{opt} - \min\{\pi_l, \pi_{2k-1}, \pi_{2k}\}\\
&= \widehat{Z}_{opt} - \min\left\{\min_{1\leq j\leq 2k-2}\{\pi_j\}, \pi_{2k-1}, \pi_{2k}\right\}
\end{aligned}$$

Thus

$$\max_{1\leq j\leq 2k}\left\{\widehat{Z}_j\right\} = \widehat{Z}_{opt} - \min_{1\leq j\leq 2k}\{\pi_j\} \tag{14}$$

Then, it easy to see now that the branch whose penalty is the smallest, gives the greatest value of the objective function on all pendant vertices of the rooted tree. ■

Theorem 2 *The optimal solution of problem $(\mathcal{P})$ is obtained by the branch and bound algorithm in a finite number of iterations.*

Proof
Let $x^l = (x_i)$, $i \in I_B$, the optimal continuous solution of problem $(\mathcal{P}_1)$ and let x^l_{kt} a non integer component of x^l for some $kt \in L \cap I_B$. Since the feasible solution of $(\mathcal{P})$ is required to be integer, we implicitly impose either $x^l_{kt} \leq \lfloor c_{kt} \rfloor$ or $x^l_{kt} \geq \lceil c_{kt} \rceil$ to corresponding solution to problem $(\mathcal{P})$. Then the new branch is constructed by addition of one of the above. Therefore, the convergence of the algorithm in a finite number of iterations is assured by the boundedness of S. If S is not bounded, the maximum of the objective function is not attained but we can compute the supremum given by equation (4) of the objective function (if it exists) by solving the relaxed problem $(\mathcal{P}_1)$. ■

4 An illustrative example

Consider the mixed integer linear fractional problem $(\mathcal{P})$ given by :

$$\begin{aligned}
&\text{Maximize } Z = \frac{3x_1 - 2x_2 + x_3}{2x_1 + 3x_2 + 1} \\
&\text{subject to} \\
&\qquad 6x_1 - 4x_2 + 8x_3 \leq 15, \\
&\qquad 2x_1 + x_2 - 2x_3 \leq 7, \\
&\qquad 2x_1 + 2x_2 + x_3 \leq 9, \\
&\qquad x_1, x_2, x_3 \geq 0 \textit{ and } x_1, x_2 \textit{ integers}
\end{aligned}$$

We solve the relaxed problem $(\mathcal{P}_1)$. The optimal simplex tableau is presented as:

$$\begin{aligned}
&\text{Maximize } Z = \frac{15/2 + 3x_3 + 1/2x_4}{6 - 13/3x_2 + 7/3x_3 - 1/3x_4} \\
&\text{subject to} \\
&\qquad x_1 - 2/3x_2 + 4/3x_3 + 1/6x_4 = 5/2, \\
&\qquad x_5 + 7/3x_2 - 14/3x_3 - 1/3x_4 = 2, \\
&\qquad x_6 + 10/3x_2 - 5/3x_3 - 1/3x_4 = 4, \\
&\qquad x_j \geq 0,\ j = 1, 2, \ldots, 6
\end{aligned}$$

The optimal continuous solution is $x_1 = 5/2$, $x_5 = 2$, $x_6 = 4$ and $x_j = 0$ otherwise. Since $L = \{1, 2\}$, $I_B = \{1, 5, 6\}$ then $L \cap I_B = \{1\}$. the solution $x_1 = 5/2$ is not feasible to problem $(\mathcal{P})$. Then, we compute the penalties π_1 and π_2 of the added constraints $x_1 \leq \lfloor 5/2 \rfloor$ and $x_1 \geq \lceil 5/2 \rceil$, respectively, and we find $\pi_1 = 1/20$ and $\pi_2 = 65/148$. we select the branch whose the penalty is the smallest and augment the respective constraint to problem $(\mathcal{P}'_1)$.

The obtained solution is $x_1 = 2$, $x_5 = 3$, $x_6 = 5$, $x_4 = 3$ and $x_j = 0$ otherwise. Therefore, $L \cap I_B = \{1\}$ and $x_1 = 2$, $x_2 = 0$ is an optimal feasible solution to problem $(\mathcal{P})$ with the value of Z equal to $6/5$.

5 Conclusion

In this paper, we have proposed an algorithm for solving a mixed integer linear fractional program which can be viewed as an alternative to methods proposed by Granot based on cutting plane techniques. Seshan and Tikekar's based on parametric approaches and Chandra and Chandramohan's based on a branch and bound technique via Charnes and Cooper transformation.

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