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Bipolar ranking from pairwise fuzzy outrankings

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Abstract

In this paper we propose to apply the concept of $\mathcal{L}$-valued kernels (see Bisdorff & Roubens [3, 4]) to the problem of constructing a global ranking from a pairwise $\mathcal{L}$-valued outranking relation defined on a set of decision alternatives as encountered in the fuzzy preference modelling context (see Roy & Bouyssou [11] for instance). Our approach is based on a repetitive selection of best and worst candidates from sharpest $\mathcal{L}$-valued or most credible initial and terminal kernels (see Bisdorff [6]). A practical illustration will concern the global ranking of movies from individual evaluations by a given set of movie critics.

1 Introduction

In this paper we propose to apply the concept of $\mathcal{L}$-valued kernels (see Bisdorff & Roubens [3, 4]) to the problem of constructing a global ranking from a pairwise $\mathcal{L}$-valued binary outranking relation defined on a set of decision alternatives as encountered in the fuzzy preference modelling context (see Fodor & Roubens[7] or Roy & Bouyssou [11] for instance).

First we discuss the practical problem which concerns the construction of a global ranking of movies based on individual evaluations from a set of given movie critics. In the second part of the paper, we briefly introduce the concepts of initial and terminal $\mathcal{L}$-valued kernels and show their eventual use in implementing an $\mathcal{L}$-valued ranking procedure. In the third section, we illustrate and discuss our ranking approach with the help of the results obtained on the set of movies.

2 Ranking movies from the best to the worst

In this section we first present the practical ranking problem that we propose for our investigation. In a second part, we introduce an Electre based construction of a global outranking index between alternatives to be considered(see [11]). Unfortunately our data normally contains a high rate of missing evaluations. Therefore we propose in a last subsection an innovative method for dealing with this problem within the scope of the Electre methods.

2.1 The movie critics in Luxembourg

The Luxembourg movie magazine "Graffiti" publishes every month a list of appreciations of currently shown movies in the Luxembourg movie theatres by some well known local journalists and cinema critics (see Table 1). The evaluation data set we use in this paper is collected from the July/August 1998 issue of the Graffiti magazine (the complete data set is shown in Figure 1 below). In the extract shown in Table 2 below, one may notice that the critics express their opinions on the basis of an ordinal preference scale ranging from four stars (****) (very much appreciated) to two zeros (oo) (very much disliked). A slash (/) indicates missing data, i.e. a critic did not evaluate a movie. Unfortunately, missing evaluations are rather common and we will propose below an original method for dealing with this uncertainty. In order to clearly separate the positive stars from the negative zeros, we furthermore introduce a neutral null point as separator between positive stars and negative o's, i.e. we extend the original scale to a set of seven ordinal grades $\{-2, -1, 0, 1, 2, 3, 4\}$. For an individual critic, this preference

Table 1: The Luxembourg Movie Critics in our data set

Identifier	Name	Press affiliation
jpt	JP Thilges	Revue & Graffiti
as	Alain Stevenart	La Meuse
mr	Martine Reuter	Tageblatt & RTL Radio Lëtzebuerg
dr	Duncan Roberts	Luxembourg News
pf	Peter Feist	Grengespoun
vt	Viviane Thill	Le Jeudi
jh	Joy Hoffmann	Zinemag
rei	Raoul Reis	Noticias & Radio Ara,
rr	Romain Roll	Zeitung
cs	Christian Spielman	Journal
h7	Rédaction Cinéma	Radio 100.7

Table 2: The Movie Critics' opinions in Luxembourg

Movies	*jpt*	*mr*	*vt*	*jh*	...
Kundum	****	*	*	*	...
Liar	**	**	**	***	...
The Wedding Singer	**	o	o	oo	...
The Magic Sword	**	/	/	*	...
...	...	...	...	...	...

scale gives a complete ordering $\geq$ from the best (**** = 4) to the worst (00 = -2) evaluation. For instance, critic *jpt* certainly accepts the movie *Kundum* as being at least as good as the movie *Liar*, but not the reverse. On the contrary, critic *mr* expresses just the opposite opinion.

2.2 Constructing a global outranking index

Following the general Electre methodology (see Roy & Bouyssou [11])[1], we may additively aggregate the individual outranking relations which we observe from the evaluation table by considering each of the eleven critics as an independent criterion associated with a weight of

[1] In fact, we only take into account the concordance part of the Electre methods. The discordance part is irrelevant within the scope of our discussion here.

Movies	jpt	as	mr	dr	pf	vt	jh	rel	rr	cs	h7
Abre los Ojos (ao)	**	***	**	**	**	/	**	***	***	*	/
Amantes (am)	**	**	*	/	/	o	**	**	/	**	/
American Werewolf in Paris (aw)	*	*	*	*	*	o	*	*	***	*	*
La Buena Estrella (be)	**	/	**	/	***	**	***	***	**	/	/
La Buena Vida (bv)	**	/	**	/	***	**	***	**	/	*	/
Caricies (c)	**	/	/	/	/	/	*	*	/	**	/
Deep Rising (dr)	*	/	/	*	/	/	*	o	**	*	*
En la puta calle (epc)	**	**	/	/	/	**	**	*	**	**	/
Fairy Tale, a true Story (ft)	***	/	***	**	/	/	/	/	***	*	*
Flamenco (fl)	***	/	*	**	**	/	/	**	***	*	**
Gingerbread Man (gm)	**	o	**	*	*	*	**	**	**	**	*
Hola, estas Sola? (hes)	/	***	*	/	/	/	***	/	*	/	/
Kundun (k)	****	****	*	***	**	*	*	*	*	*	**
Liar (l)	**	*	**	**	/	**	***	**	***	*	****
Love!Valour!Compassion! (lvc)	**	/	/	*	***	*	o	/	/	**	**
The Magic Sword (ms)	**	/	/	/	/	/	*	/	**	*	/
La Mirada del otro (mo)	**	/	*	**	*	*	**	*	**	**	/
Paparazzi (pp)	*	*	**	/	*	*	**	**	*	**	*
La Pasion Turca (pt)	**	/	*	/	/	*	*	/	/	**	/
Perdita Durango (pd)	/	***	*	***	**	o	*	*	****	/	**
Primary Colours (pc)	***	**	**	**	/	**	**	**	*	*	/
Secretos del Corazon (scd)	***	**	**	/	***	***	***	***	***	***	/
Serial Lover (sl)	**	***	**	/	/	**	***	/	/	**	**
Swept from the Sea (ss)	*	/	/	**	**	/	o	o	/	*	*
TerritioComanche (tc)	o	/	/	**	*	/	**	**	*	**	/
A Thousand Acres (ta)	**	*	*	**	***	*	/	/	/	/	/
Vertigo 70mm (v)	****	****	***	****	****	****	***	****	****	****	****
The Wedding Singer (ws)	**	**	o	*	/	o	oo	**	**	/	*
Wings of the Dove (wd)	***	*	***	****	**	**	**	o	**	*	***

Figure 1: The complete evaluation data (Source: Graffiti July/August 1998)

1/11.

In general, let M denote the set of considered movies and for each $m \in M$, let C_m be the subset of critics who have expressed their opinions about movie m. For each movie $m_i \in M$ and critic $c \in C_{m_i}$, let $m_i(c)$ denote the evaluation the critic has expressed. A natural outranking index s_{ij} logically evaluating the proposition "*movie m_i is evaluated as at least as good as movie m_j*" may be computed in the following way:

$$s_{ij} = \frac{\mid \{c \in C_{m_i} \cap C_{m_j} \quad : \quad m_i(c) \; \geq \; m_j(c)\} \mid}{\mid C_{m_i} \cap C_{m_j} \mid} \qquad (1)$$

We may see in s_{ij} the result of a voting in favour of the proposition "*movie m_i is evaluated as at least as good as movie m_j*" and we could take such a proposition as logically verified if it is supported by at least a majority of critics. In Table 3 we can see the resulting global outranking index on the illustrative sample given in Table 2 above.

Unfortunately, the given evaluation data frequently contains missing values, namely whenever a critic has not had the opportunity to see and/or to express his opinion about a movie

Table 3: The global outranking index s_ij

Movies	k	l	ws	ms	...
Kundum (k)	–	**.40**	.78	.75	...
Liar (l)	.70	–	.90	.100	...
The Wedding Singer (ws)	.22	.40	–	.75	...
The Magic Sword (ms)	**.75**	.60	.100	–	...
...	...	...	...	...	...

on the given evaluation list (see Figure 1 above).

2.3 Taking into account missing evaluations

Our idea here is that two movies both of which have not been seen by a critic may not be ranked, i.e. the credibility of the proposition that "*the first movie is considered by the critic at least as good as the second movie*" must admit the $\mathcal{L}$-undetermined value, i.e. the negational fix-point $\frac{1}{2}$(see [4]).

Now the more evaluations by a critic are missing for a particular movie, the more its global outranking relation wrt to all the other's will tend to the $\mathcal{L}$-undetermined value $\frac{1}{2}$.

Formally, we adjust the above outranking index (see equation 1) in the following way. Let s_{ij} be the original outranking index computed from the evaluations of movies m_i and m_j and let m_{ij} be the ratio of common evaluations wrt to the number of possible critics. Then the proposed adjusted outranking index s_{ij}^m is defined in the following way:

$$s_{ij}^m \;=\; m_{ij} \cdot s_{ij} \;+\; (1 - m_{ij}) \cdot \tfrac{1}{2} \qquad (2)$$

A graphical representation of the transformation may be seen in Figure 2.

The resulting adjusted outranking index is shown in Table 4.

Semantically speaking, we adjust the outranking index by adding half of the relatively missing evaluations as outranking and the other half as not outranking propositions. In the limit, if m_{ij} approaches 1 (both movies have been seen by nearly all critics), s_{ij} remains more or less unchanged. This is observed for instance if we consider the movies *Kundum* and *Liar*, where $m_{k,l} = 10/11$, $s_{k,l} = .40$ and $s_{k,l}^m = .41$. On the other hand, if m_{ij} approaches the value 0, (no common evaluations), s_{ij} is more and more restricted to close values around $\frac{1}{2}$. In case a small number of critics largely prefers a movie to another one, this local preference is always

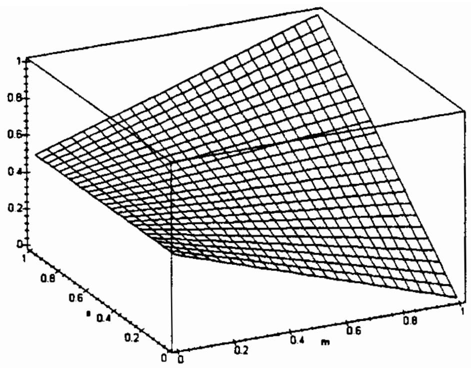

Figure 2: Taking into account missing evaluations

Table 4: The adjusted global outranking index s^m_{ij}

Movies	k	l	ws	ms	...
Kundum (k)	–	**.41**	.73	.59	...
Liar (l)	.68	–	.86	.73	...
The Wedding Singer (ws)	.27	.41	–	.59	...
The Magic Sword (ms)	**.59**	.55	.68	–	...
...	...	...	...	...	...

transformed into an $\mathcal{L}$-true global outranking but the smaller the number of actually voting critics is, the higher the $\mathcal{L}$-undeterminedness will tend to be.[2] This case is observed for instance when comparing the movies *Kundum* and *The Magic Sword* where $m_{k,ms} = 4/11$, $s_{k,ms} = .75$, and $s^m_{k,ms} = .59$.

The result of our construction finally gives an aggregate $\mathcal{L}$-valued pairwise outranking relation on the set of all 29 movies under consideration (see Figure 3). On the basis of this fuzzy outranking relation, we would now like to construct a global ranking of the movies from the best to the worst evaluated ones.

[2]In fact, the simple majority principle for asserting an outranking situation is not restricted by any required minimal quorum of effectively given evaluations.

Movies	Abre los Ojos	Amantes	American Werewolf in Paris	La Buena Estrella	La Buena Vida	Caricies	Deep Rising	En la puta calle	Fairy Tale, a true Story	Flamenco	Gingerbread Man	Hola, estas Sola?	Kundun	Liar	LoveValourCompassion	The Magic Sword	La Mirada del otro	Paparazzi	La Pasion Turca	Perdita Durango	Primary Colours	Secretos del Corazon	Serial Lover	Swept from the Sea	TerritorioComanche	A Thousand Acres	Vertigo 70mm	The Wedding Singer	Wings of the Dove
Abre los Ojos	-	68	91	59	59	59	77	68	55	73	82	59	64	77	55	68	77	77	59	64	77	50	55	77	73	64	9	82	64
Amantes	50	-	82	36	50	68	68	68	45	59	64	45	55	55	59	64	68	64	64	64	55	27	41	68	68	59	18	77	55
American Werewolf in P	27	36	-	27	27	50	82	36	50	41	45	50	45	32	36	59	45	50	45	45	27	18	18	64	45	41	0	64	36
La Buena Estrella	68	73	73	-	77	64	68	73	36	55	82	64	73	68	68	64	82	82	68	68	68	55	68	68	73	68	27	77	64
La Buena Vida	68	68	82	68	-	59	68	64	45	64	73	59	73	77	64	64	73	73	64	73	68	45	64	73	64	68	27	73	64
Caricies	50	50	68	45	50	-	68	59	50	45	50	45	59	50	64	64	59	50	64	59	41	32	55	68	50	55	32	55	50
Deep Rising	32	32	64	41	41	41	-	36	45	32	45	50	45	27	45	59	32	50	45	36	41	27	32	68	41	41	18	59	45
En la puta calle	50	68	73	55	55	68	73	-	45	41	73	45	64	55	68	68	77	73	68	55	64	27	55	68	64	64	18	68	732
Fairy Tale, a true Story	73	55	77	64	64	50	73	55	-	68	68	59	50	68	50	64	64	64	55	41	73	59	50	68	59	64	32	73	59
Flamenco	64	59	86	45	55	55	77	59	68	-	68	59	68	64	55	64	73	64	55	59	68	41	50	77	68	59	14	77	59
Gingerbread Man	45	73	82	45	55	68	82	64	41	41	-	50	55	41	64	68	82	86	73	45	55	18	45	64	73	50	0	82	36
Hola, estas Sola?	50	64	59	45	50	55	50	55	41	50	50	-	59	50	55	50	55	59	59	59	59	50	55	55	59	59	41	59	50
Kundun	55	55	91	27	36	59	73	45	59	68	55	59	-	41	64	59	64	59	64	82	55	27	45	82	55	68	18	73	45
Liar	68	64	95	68	77	59	82	64	59	73	86	59	68	-	68	68	77	82	64	59	73	41	64	77	86	73	23	82	68
LoveValourCompassion	55	59	73	50	55	55	64	50	50	55	73	45	55	41	-	55	59	68	59	55	36	36	55	68	55	59	18	73	36
The Magic Sword	50	45	59	55	55	55	68	50	45	45	50	50	59	50	55	-	50	50	55	50	50	32	45	64	50	55	32	64	50
La Mirada del otro	50	68	82	36	36	68	77	68	45	45	73	55	64	41	68	68	-	68	73	55	50	14	45	68	73	64	9	73	45
Paparazzi	41	64	86	27	45	59	68	45	45	45	77	50	59	45	50	50	68	-	64	50	59	18	36	68	77	55	5	59	41
La Pasion Turca	50	64	73	41	45	64	64	50	45	55	55	50	64	45	68	64	64	55	-	64	36	27	45	64	55	64	27	68	36
Perdita Durango	55	55	91	32	27	59	73	55	59	68	55	59	73	41	55	59	64	50	55	-	45	36	45	73	55	55	18	77	45
Primary Colours	59	73	82	50	68	59	68	64	55	68	73	50	64	73	64	59	68	77	64	55	-	41	50	73	68	73	9	77	64
Secretos del Corazon	77	82	91	82	82	68	73	82	59	77	91	59	73	86	73	68	86	91	73	64	86	-	68	73	77	73	18	82	82
Serial Lover	73	77	82	68	73	64	68	73	50	59	82	64	64	73	73	64	73	82	73	73	68	50	-	68	64	68	27	77	55
Swept from the Sea	50	32	64	32	36	32	68	32	59	50	45	45	36	41	41	45	41	50	36	36	45	27	32	-	50	45	18	55	45
TerritorioComanche	45	59	64	27	45	59	59	55	50	50	64	50	55	50	55	50	64	68	55	45	68	23	45	59	-	45	18	55	45
A Thousand Acres	55	59	77	50	50	55	59	45	45	59	68	50	50	55	68	55	73	64	64	55	36	36	41	64	64	-	23	64	41
Vertigo 70mm	91	82	100	82	82	68	82	82	77	86	100	68	100	95	82	68	91	95	73	91	91	91	82	82	82	77	-	91	100
The Wedding Singer	27	59	64	41	45	55	68	59	36	32	64	41	27	36	45	55	45	59	41	59	41	27	32	55	55	45	9	-	36
Wings of the Dove	64	55	82	55	55	50	82	55	68	68	82	50	73	59	64	68	73	77	64	64	73	27	55	82	64	68	18	73	-

Figure 3: Global outranking index (s^m_{ij})

3 Ranking by repetitive best and worst choices

In this section, we first introduce the concepts of initial and terminal $\mathcal{L}$-valued kernels (see Bisdorff[6]) for implementing a best and/or worst choice procedure from a pairwise $\mathcal{L}$-valued based outranking index. In a second part we then show how a recursive use of this approach allows us to generate a global ranking.

3.1 Initial and terminal kernels

Let $G(A, R)$ be a simple graph with R being a crisp binary relation on a finite set A of dimension n. A subset Y of A is an initial (dominant), respectively terminal (absorbent) kernel of the graph G, if it verifies conjointly the following right and left interior stability condition:

$$\forall a, b \in A(a \neq b) : (aRb)(\text{respectively } (bRa)) \wedge (b \in Y) \Rightarrow (a \notin Y) \tag{3}$$

and respective exterior stability condition:

$$\forall a \in A : (a \notin Y) \Rightarrow (\exists b \in A : (b \in Y)) \wedge (bRa)(\text{respectivlely } (aRb)) \tag{4}$$

Interior stability simply reqires that all kernel members are mutually incomparable and exterior initial (respectively terminal) stability requires that each alternative not selected in a kernel must be dominated by (respectively dominates) at least one kernel member.

Terminal kernels on simple graphs were originally introduced by J. Von Neumann and O. Morgenstern ([8]) under the name 'game solution' in the context of game theory. J. Riguet ([9]) proposed the name 'noyau (kernel)' for Von Neumann's 'game solution' and B. Roy ([10]) introduced the reversed terminal or initial kernel construction as possible dominant choice procedure in the context of his multicriteria Electre decision methods. Terminal kernels were studied by C. Berge ([1, 2]) in the context of the Nim game modelling. The crisp kernel concept may be generalized to our $\mathcal{L}$-valued preference modelling context (see appendix A). For instance for the small example of Table 4, we would obtain the following results: *Liar* (l) and *The Magic Sword* (ms) appear as credible initial kernel solutions and *The Wedding Singer* (ws) and again *The Magic Sword* (ms) appear as credible terminal kernel solutions.

Table 5: Initial and terminal kernels from pairwise outranking index

Movies	k	l	ws	ms
K_1^i	.32	**.68**	.32	.32
K_2^i	.45	.45	.45	**.55**
K_1^t	.32	.32	**.68**	.32
K_2^t	.41	.41	.41	**.59**

For $\mathcal{L}$-un-cyclic graphs, i.e. $\mathcal{L}$-valued graphs not containing any $\mathcal{L}$-true supported circuit, $\mathcal{L}$-valued initial and terminal kernel solutions are unique, and recursive elimination of best and worst choices makes apparent the underlying transitive $\mathcal{L}$-valued ordering of the alternatives. But in general, as in our example above, we may observe several admissible initial as well as terminal $\mathcal{L}$-valued kernel solutions. Therefore we introduce a special ordering on $\mathcal{L}$-valued kernel solutions which is inspired by the concept of distributional dominance as used in the context of stochastic dominance.

Let $K = \{K_1, K_2, \ldots, K_k\}$ be a set of kernel solutions defined on a given $\mathcal{L}$-valued graph $G^{\mathcal{L}} = (A, R)$ where the set A contains a finite number n of alternatives. We say that a kernel solution K_i *is at least as credible as* a kernel solution K_j iff the cumulative frequencies of $\mathcal{L}$-true values of K_i are all shifted towards truth value 1 (certainly true) if compared to the cumulative frequencies of $\mathcal{L}$-true values of K_j and, vice versa, the cumulative frequencies of $\mathcal{L}$-false values of K_i are all shifted towards the truth value 0 (certainly false) if compared to the cumulative frequencies of $\mathcal{L}$-false values of K_j. From the resulting most credible initial kernel solutions we extract all maximal dominating alternatives as best choices and similarly, from the most credible terminal kernel solutions, we extract the maximal dominated alternatives as worst choices.

In our example we would keep as most credible initial kernel solution the first solution with *Liar* (l) selected as best choice, and *The Wedding Singer* (ws) as most credible terminal kernel solution, i.e. as worst choice.

In the case where no non trivial, i.e. not $\mathcal{L}$-undetermined, kernel solutions exist we exhibit an unrankable residue. The earlier an alternative is selected as best or worst, the more reliable this choice is, such that the eventual last unrankable residual class appears as the least credible result of all.

We have now introduced all ingredients to our bipolar ranking procedure.

3.2 The bi-pole ranking algorithm

The main step of the procedure consists in a recursive computing of initial and terminal $\mathcal{L}$-valued kernels solutions on successive restrictions of the original graph by removing the alternatives corresponding to $\mathcal{L}$-valued disjunction of the Ith most credible kernels in the sense of the above introduced first order credibility dominance (see appendix B). The complexity of the kernel computation is theoretically in $\mathcal{O}(3^n)$ with n the dimension of set A, but efficient concurrent finite domains enumeration techniques in a constraint logic programming environment allow us to solve problems up to 50 or even 60 alternatives (see[5]).

On the small sample of four movies of Table 2, we obtain the following results when considering the global outranking index shown in Table 4 above:

```
Bi-pole ranking of relation : Table 2
choices :
1st step: action set A = {k, l, ws, ms}
Ki ={k(32), l(68), ws(32), ms(32)} %% most credible initial kernel solution
1st best choice  : {l}(68)
Kt ={k(32), l(32), ws(68), ms(32)} %% most credible terminal kernel solution
1st worst choice : {ws}(68)
   2nd step: action set A = {k, ms}
   Ki ={k(73), ms(27)} %% most credible initial kernel solution
   2nd best choice  : {k}(73)
   Kt ={k(27), ms(73)} %% most credible terminal kernel solution
   2nd worst choice  : {ms}(73)
      3rd step: action set A = {}
```

```
residual class : {}(50)
```

Among the four movies, *Liar* (l) appears as 1st best choice with credibility 68% and *The Wedding Singer* (ws) as 1st worst choice with the same credibility. Second best (resp. worst) choice gives *Kundum* (k) (resp. *The Magic Sword* (ms)). The eventual unrankable middle class is empty in this example.

To further illustrate our approach we now solve the complete ranking problem.

4 Global ranking of all movies

In a first part, we show the outcome of our algorithm on the complete data set and in a second part we discuss some methodological considerations with respect to our bipolar ranking approach and our treatment of missing values.

4.1 Bipolar ranking results

The outcome of our bipolar ranking procedure is the following:

```
Bi-pole ranking of relation : Complete data set
1st best  : Vertigo 70mm (v)(68)
  2nd best   : Secretos del Corazon (csd) (59)
    3rd best   : Liar (l) (59)
      4th best   : Abre los Ojos (ao),  (55)
         residual class :
         Amantes (am), La Buena Estrella (be), La Buena Vida (bv),
                 Caricies (c), Deep Rising (dr), En la puta calle (epc),
                 Fairy Tale, a true story (ft), Flamenco (fl),
                 Gingerbread Man (gm), Hola, esta sola? (hes),
                 Kundum (k), Love!Valour!Compassion! (lvc),
                 La Mirada del otro (mo), The Magic Sword (ms),
                 Paparazzi (pp), Perdita Durango (pd), La Pasion Turca (pt),
                 Primary Colours (pc), Serial Lover (sl),
                 A Thousand Acres (ta), TeritioCommanche (tc),
                 Wings of the Dove (wd)(50)
  2nd worst  : Swept from the Sea (ss), The Wedding Singer (ws)(55)
1st worst : American Werewolf in Paris (aw)(59)
```

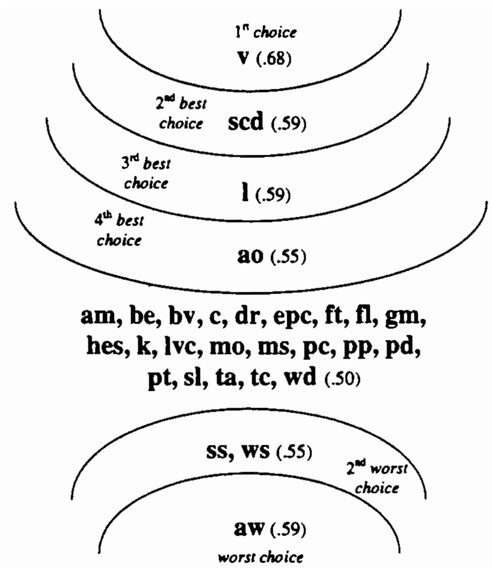

Figure 4: Best against worst choices

The ranking result can be graphically represented as in Figure 4.

The movie *Vertigo 70 mm* (**v**), a recent 70mm restoration of a Hitchcock classic appears as global winner with a credibility of 68%. This result is not surprising as its evaluations are unanimously very high with 9×'****' and 2×'***' evaluations. Second selected is *Secretos del Corazon* (**scd**) with 7×'***' and 2×'**' evaluations (see Figure 1).

On the other hand, one movie is immediately designated as worst evaluated: *American Werewolf in Paris* (**aw**) with 1×'***', 9×'*' and 1×'o' evaluations.

If we sort the rows of our complete data set on the rank obtained through our bipolar ranking procedure, we obtain the list shown in Figure 5.

4.2 Methodological discussion

4.2.1 Non-independence with respect to the relevant set of alternatives

Reconsidering our illustrative sample, we may notice that *Liar* (**l**) is indeed ranked before *Kundum* (**k**) and *The Magic Sword* (**ms**), which appear unranked in the residual unrankable class, whereas *The Wedding Singer* (**ws**) is designated as worst choice against all three others. This fact reminds us that we must consider our bipolar ranking result as immediately related to

Movies	jpt	as	mr	dr	pf	vt	jh	rel	rr	cs	h7	rank
Vertigo 70mm	****	****	***	****	****	****	***	****	****	****	****	1
Secretos del Corazon	***	**	**	/	***	***	***	***	***	***	/	2
Liar	**	*	**	**	/	**	***	**	***	*	****	3
Abre los Ojos	**	***	**	**	**	/	**	***	***	*	/	4
Amantes	**	**	*	/	/	o	**	**	/	**	/	5
La Buena Estrella	**	/	**	/	***	**	***	***	**	/	/	5
La Buena Vida	**	/	**	/	***	**	***	**	/	*	/	5
Caricies	**	/	/	/	/	/	*	*	/	**	/	5
Deep Rising	*	/	/	*	/	/	*	o	**	*	*	5
En la puta calle	**	**	/	/	/	**	**	*	**	**	/	5
Flamenco	***	/	*	**	**	/	/	**	***	*	**	5
Fairy Tale, a true Story	***	/	***	**	/	/	/	/	***	*	*	5
Gingerbread Man	**	o	**	*	*	*	**	**	**	**	*	5
Hola, estas Sola?	/	***	*	/	/	/	***	/	*	/	/	5
Kundun	****	****	*	***	**	*	*	*	*	*	**	5
Love!Valour!Compassion!	**	/	/	*	***	*	o	/	/	**	**	5
La Mirada del otro	**	/	*	**	*	*	**	*	**	**	/	5
The Magic Sword	**	/	/	/	/	/	*	/	**	*	/	5
Primary Colours	***	**	**	**	/	**	**	**	*	*	/	5
Perdita Durango	/	***	*	***	**	o	*	*	****	/	**	5
Paparazzi	*	*	**	/	*	*	**	**	*	**	*	5
La Pasion Turca	**	/	*	/	/	*	*	/	/	**	/	5
Serial Lover	**	***	**	/	/	**	***	/	/	**	**	5
A Thousand Acres	**	*	*	**	***	*	/	/	/	/	/	5
TerritioComanche	o	/	/	**	*	/	**	**	*	**	/	5
Wings of the Dove	***	*	***	****	**	**	**	o	**	*	***	5
Swept from the Sea	*	/	/	**	**	/	o	o	/	*	*	6
The Wedding Singer	**	**	o	*	/	o	oo	**	**	/	*	6
American Werewolf in Paris	*	*	*	*	*	o	*	*	***	*	*	7

Figure 5: Final ranking of the movies

the actually considered set of alternatives. A same couple of alternatives, especially appearing near the unrankable middle class, may very well undergo profound and contradictory ranking variations if considered with different reference alternatives, particularly if missing evaluations are involved. This problem may become critical with certain applications, but in our case, as the considered reference set is independently defined by the editor of the 'Graffiti' magazine, we are not really concerned.

4.2.2 Partial versus complete ranking

A second practical problem is constituted by the rather large unrankable (somehow equivalent) middle class that we obtain. This result depends to some extent on the high rate of missing evaluations which introduce a considerable part of $\mathcal{L}$-undeterminedness into our adjusted global outranking index. But it also indicates the existence of contradictory evaluations as observed for instance about *The Wings of the Dove* with 1×'****', 3×'***', 4×'**', 2×'*' and even 1×'0'. Such evaluations reduce the credibility of a refined global ranking. In this case, the critics express highly divergent opinions which make it difficult to situate this movie against all the others. The size of the residual middle class thus gives a hint towards the existence of either missing values or the presence of contradictory evaluations. In our opinion, this prudent ranking approach, which in the final result keeps traces of contradictory as well as missing evaluations, constitutes precisely the strength of our use of recursive initial and terminal kernels extraction technique.

4.2.3 Other methods for dealing with missing evaluations

A third practical problem obviously concerns the treatment of missing evaluations. An alternative approach would consist in replacing missing evaluations by the neutral point on the preference scale, i.e. the separator between stars and zeros. In our case, this approach indeed largely reduces the size of the unrankable middle class and selects with certainty *Vertigo 70mm* as first best choice, but the worst choices are somehow changed and the result is less convincing. Indeed, in our data set, one star '*' evaluations appear as already very weak evaluations, and artificially adding a lot of even lower evaluations in replacement for the missing ones modifies quite a lot the original bottom ranking results (compare Figures 6 and 4).

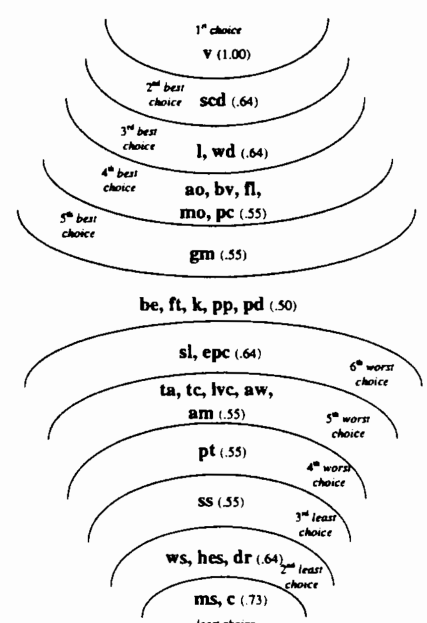

Figure 6: Replacing missing values with a neutral evaluation

Yet another and classic idea therefore consists in replacing missing evaluations with a mean evaluation from all observed evaluations in the row. For our data set, the resulting complete bipolar ranking appears more or less compatible with the one we obtain, except that higher credibilities are generally associated with the results and the residual unrankable middle class is reduced to only three items. Unfortunately, the greater precision is artificially introduced and is not originally supported by the observed data. To appreciate the difference in results, we may notice that the evidently best choice, i.e. *Vertigo 70mm* (v), is selected under this approach with certainty (100%), whereas it is only supported by a credibility of 68% in our approach. This increase in uncertainty is induced by our explicit consideration of the rather large part of missing evaluations.

5 Conclusion

In this paper, we have introduced an innovative bipolar ranking approach based on the concepts of initial and terminal kernel solutions from a pairwise $\mathcal{L}$-valued comparison index. We have illustrated our approach with the help of a real-size ranking problem of movies on the basis of a set of evaluations by known movie critics. Finally, an original method for dealing with numerous missing evaluations has been developed and discussed.

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Appendices

A $\mathcal{L}$-valued generalization of initial and terminal kernels

Let $G_{\mathcal{L}} = (A, R)$ be a simple $\mathcal{L}$-valued graph with R being a binary relation on a set A of decision alternatives. The relation R is logically evaluated in a symmetric credibility domain $\mathcal{L} = \{V, \leq, \min, \max, \neg, \rightarrow, 0, \frac{1}{2}, 1\}$ (see [3]), where V is a finite set of $2m+1$ rational values between 0 and 1 with min and max as conjunctive and disjunctive operators. The negation operator '$\neg$' in V implements a strict anti-tonic bijection giving $\frac{1}{2}$ as negational fix-point and the implication operator '$\rightarrow$' verifies the following condition: $\forall u, v \in V : (u \leq v) \Leftrightarrow (u \rightarrow v) = 1$.

All degrees of credibility $v \in V$ such that $v > \frac{1}{2}$, are denoted as being *$\mathcal{L}$-true*, i.e. supporting more the truthfulness than the falseness of a relational proposition and all degrees $v < \frac{1}{2}$ are denoted as being *$\mathcal{L}$-false*, i.e. supporting more the falseness than the truthfulness of a given relational proposition. The median truth value $\frac{1}{2}$ appears as logically undetermined and therefore expresses most uncertainty towards truthfulness or falseness of a given relational proposition.

Let $\{k_R\}$ be a singleton set. We assume Y to be an $\mathcal{L}$-valued binary relation defined on $A \times \{k_R\}$, i.e. a function $Y : A \times \{k_R\} \rightarrow V$, where each $Y(a, k_R), \forall a \in A$, is supposed to indicate the degree of credibility of the proposition that 'element a is included in the kernel k_R'. As k_R is a constant, we will simplify our notation by dropping the second argument and in the sequel $Y(a), \forall a \in A$, denotes an $\mathcal{L}$-valued characteristic vector for the kernel membership function defined from a given relation R.

As degrees of credibility of the propositions that 'a is a right (respectively left) interior stable element of A' we choose a value $Y(a)$ verifying the following conditions:

$$\max_{b \in A, (a \neq b)} [\min(aRb), Y(b))] \;\; \rightarrow \;\; \neg Y(a) = 1 \tag{5}$$

$$\max_{b \in A, (a \neq b)} [\min((aR^{-1}b), Y(b))] \;\; \rightarrow \;\; \neg Y(a) = 1 \tag{6}$$

where $\neg Y$ represents the $\mathcal{L}$-negation of Y. And similarly, as degrees of credibility $Y(a)$ of the propositions that 'a is an initial (respectively terminal) stable element of A' we choose a value $Y(a)$ verifying the following respective condition:

$$\max_{b \in A, (a \neq b)} [\min(aRb), Y(b))] \;\; \leftarrow \;\; \neg Y(a) = 1 \tag{7}$$

$$\max_{b \in A, (a \neq b)} [\min((aR^{-1}b), Y(b))] \;\; \leftarrow \;\; \neg Y(a) = 1 \tag{8}$$

It is worth noticing that these conditions may be expressed in a synthetical way with the help of relational $\mathcal{L}$-valued products and inequations.

$$Y \text{is right interior stable} \;\; \Leftrightarrow \;\; R \circ Y \leq Y \tag{9}$$

$$Y \text{is left interior stable} \;\; \Leftrightarrow \;\; R^{-1} \circ Y \leq Y \tag{10}$$

$$Y \text{is absorbent stable} \;\; \Leftrightarrow \;\; R \circ Y \geq Y \tag{11}$$

$$Y \text{is dominant stable} \;\; \Leftrightarrow \;\; R^{-1} \circ Y \geq Y \tag{12}$$

These stability inequations normally admit multiple solutions (see [3]) and we choose the most credible amongst these solutions as natural candidates for $\mathcal{L}$-valued initial or terminal kernels: Y^{rt} is a right terminal kernel if

$$Y^{rt} = \max_{Y} \{Y : (R \circ Y \leq Y) \wedge (R \circ Y \geq Y)\} \tag{13}$$

Y^{ri} is an right initial kernel if

$$Y^{ri} = \max_{Y} \{Y : (R \circ Y \leq Y) \wedge (R \circ Y \geq Y)\} \tag{14}$$

Y^{la} is a left terminal kernel if

$$Y^{lt} = \max_{Y} \{Y : (R \circ Y \leq Y) \wedge (R \circ Y \geq Y)\} \tag{15}$$

```
            output unrankable residue : A_I
            stop
        enddo
    J ← I + 1
    A_J ← A_I − (Ã_I ∪ Â_I)
    R_J ← restriction of R_I to A_J
    output Ith best choices : Ã_I
    output bipoleranking (G_J = (A_J, R_J))
    output Ith worst choice : Â_I
  enddo
endbipoleranking
```